TEACHING ARITHMETIC

in the

ELEMENTARY SCHOOL

Volume II, Intermediate Grades

by

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PREFACE

It is a common observation that those who teach arithmetic in the intermediate grades tend to follow closely the pages of the textbook which happens to be in use. There are fewer reasons to criticize this practice than there were before the development of recent improvements in arithmetic texts but it is still true that the best teaching is the result of plans which are more comprehensive than the mere following of a textbook.

One reason for this is the fact that no textbook, however well it may have been prepared, is ideally suited to the interests and needs of all pupils and groups of pupils. Individual and local adaptations frequently are required if the most desirable program is to be provided. Furthermore, textbooks can contain no more than brief hints as to method. Nor do they provide the teacher with basic points of view as to the contribution of arithmetic to an individual's growth and development and with an adequate theory of arithmetic instruction.

For twenty years, a considerable portion of the author's time and energy has been occupied with the professional education of teachers and prospective teachers in arithmetic. After seven years, the course for teachers in the intermediate grades had developed into a mimeographed volume of some three hundred pages. Two years later (1927) TEACHING ARITHMETIC IN THE INTERMEDIATE GRADES was published. This volume found immediate favor and has been widely and extensively used for eleven years. Plans to revise the book had not

gone far before it became apparent that a mere revision would not be sufficient. Like Volume I of this series, this book is a new book.

Since 1927, much has been written on arithmetic and arithmetic teaching in the intermediate grades. The author has reviewed thousands of pages of published materials and has undertaken to bring together here the important contributions and to make them available to the teacher in training and to the teacher in service. The reader will find that the research findings and the ideas of many persons have been used. An examination of these pages will reveal the fact that the names of no fewer than 93 persons have been mentioned.

In Chapter I, the different functions of arithmetic and the various theories of arithmetic instruction are examined and appraised. Subsequent chapters develop the subject matter of the various processes in the light of each of these functions. What has very well been called the meaning theory permeates this treatment. This theory adds its own unique contributions to the desirable features of the drill theory and the theory of incidental learning.

The author is much indebted to those whose published writings he has used. He is also under obligation to those who have been students in his own classes. Their previous educational experience has ranged all of the way from recent high school graduation to the attainment of the degree of doctor of philosophy and their educational positions have ranged from that of teacher in the humble one-room school to that of the college presidency. They number more than six thou-

sand persons. Many others, in the United States and in foreign countries, have been contacted personally and through correspondence. Letters have come from many states, from Canadian provinces and from such remote points as South Africa. The ideas gained from these sources and the frank criticisms of TEACHING ARITHMETIC IN THE INTERMEDIATE GRADES which have been offered have been very helpful in the preparation of this book.

What good this book may contain should be credited to persons other than the author for without their assistance the book could not have been produced. On the other hand, the author will assume responsibility for any recommendations with which the reader may not agree for these recommendations have been shaped in part by his own investigations, his own experiences, his own interpretations, and his own philosophy.

The author's purpose in writing this book may be stated very briefly and very simply. The book is intended to assist those who would learn to teach arithmetic, particularly in grades four, five, and six, and to indicate how those who are already teaching may teach better. It is designed for teachers, prospective teachers, supervisors, superintendents, and other students of elementary education.

There is evidence to the effect that much improvement has been accomplished in the teaching of arithmetic in recent years. But there is also abundant evidence to support the contention that only a fair beginning has been made in improving arithmetic teaching. What could be accomplished by good teachers, equipped with good textbooks and other teaching ma-

terials, and using good methods, is undoubtedly far more than is now being accomplished in our schools.

This book is offered with the hope that it may make a contribution toward improvement in arithmetic teaching.

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CONTENTS

CHAPTER 1

WHY SHOULD CHILDREN STUDY ARITHMETIC?

To many, it will seem strange that a question can be raised as to the place of arithmetic in the curriculum of the elementary school. Until recently, we have indeed tended to take arithmetic for granted and to assume that its worth justified the time and effort required to teach it. Scant attention was paid to arithmetic in the schools of colonial America, to be sure, but it came gradually to occupy a prominent place because of its obvious practical worth. Since the colonial period the fortunes of this subject have varied but not until recently has its position as one of the core subjects of the elementary school curriculum been in jeopardy.

Trends in arithmetic teaching. In the early part of the nineteenth century, arithmetic was prominently featured in the school curriculum but it was taught in a very formal manner. Little effort was made to lead pupils to understand what they did. They memorized rules and then undertook to apply these rules in the solution of examples and problems. Arithmetic was thought of as the science of computing and it was considered to be sufficient if the pupils learned well the tricks of computation.

A drastic reform was undertaken by Warren Colburn in the arithmetic textbooks which he published in 1821. Rules were abolished and an effort was made to rationalize the processes presented. Drill was introduced and oral teaching was recommended. The Colburn arithmetics were written in an effort to make the subject

more interesting to children and to lead them to understand what they learned.

The result was a marked development of arithmetic as a subject of study. It gradually demanded a larger and larger amount of time in the school program and it came to occupy a more and more prominent place in the thinking of those who planned the curriculum. In the latter half of the century, however, teaching had again become so formal that by 1890 critics were again demanding a reform. The reform took the direction of a reduction in the amount of time devoted to arithmetic and an elimination of material which could not be shown to have social utility. Today, it is a common practice to challenge the inclusion in the curriculum of any arithmetic topic or material which can not be shown to have social worth.

Standard tests in arithmetic began to appear in about the year 1913. The results of these tests revealed deficiencies which led rapidly to a marked increase in drill on the fundamental operations. The *Courtis Standard Practice Tests in Arithmetic* were published in 1914 and the *Studebaker Economy Practice Exercises in Arithmetic* in 1916. These were followed by many other drill sets and work books all of which placed emphasis upon a rather narrow computational function of arithmetic. So great has been the emphasis upon arithmetic as computation, with standards for speed and accuracy, that only in the last few years (since about 1930) has it been possible to gain a hearing for other functions of arithmetic, such as the *informational* function and the *sociological* function. At the present time (1938) there are a great many schools, probably a vast majority, in

which the teaching of arithmetic is primarily a matter of teaching computation.

Since 1930, there has been an increasing demand for a more liberal interpretation of the functions of arithmetic. It is not intended that training in computation shall be eliminated but that there shall be somewhat less emphasis upon computation and more emphasis upon the other functions of arithmetic. In 1930, the National Society for the Study of Education published the *Twenty-Ninth Yearbook,*[1] a report of its Committee on Arithmetic. In some 700 pages, this report contributes a wealth of valuable suggestions for improving the teaching of arithmetic but emphasizes largely the computational function. The Reviewing Committee, in a 29-page critique of the yearbook, indicates the importance of the informational function, the sociological function, and the psychological function, as well as the computational function.[2] These functions will be discussed more fully in later pages.

Theories of arithmetic instruction. It is suggested by Brownell[3] that there are three prominent theories of arithmetic instruction in vogue among teachers at the present time. These theories are:

1. The Drill Theory

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[1] Published by The Public School Publishing Company, Bloomington, Illinois.

[2] Op. cit., pp. 681-709.

[3] Brownell, William A. "Psychological Considerations in the Learning and Teaching of Arithmetic." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 1-31. The reader will find many points of similarity between Brownell's very excellent contribution and the following discussion.

2. The Incidental-Learning Theory
3. The "Meaning" Theory

Although very few teachers subscribe to one of these theories to the exclusion of the other two, most of the work done in a typical arithmetic class period is probably supported primarily by one or another of these three theories.

The drill theory. According to the drill theory, the learning of arithmetic consists primarily of the acquisition of a large number of specific facts and skills. These are acquired by the method of repetition. Thus, the pupil memorizes the 100 basic addition facts such as $\begin{array}{r}4\\3,\\\hline 7\end{array}$ $\begin{array}{r}3\\4,\\\hline 7\end{array}$ $\begin{array}{r}8\\6,\\\hline 14\end{array}$ $\begin{array}{r}5\\0,\\\hline 5\end{array}$ etc., and a host of higher decade addition combinations, such as $\begin{array}{r}18\\4,\\\hline 22\end{array}$ $\begin{array}{r}25\\3,\\\hline 28\end{array}$ $\begin{array}{r}46\\8,\\\hline 54\end{array}$ etc., he learns the rule for carrying, and he is drilled on these elements of addition until he can add rapidly and with "100 per cent accuracy," like a machine. There is little opportunity for him to learn the meaning of what he does. He does not *discover* that 8 and 5 are 13 but he memorizes it; he does not see 18 and 5 as related to, and an extension of, 8 and 5 but he memorizes the fact that 18 and 5 are 23; he does not see why 2 should be carried to the tens' column in the example shown but he carries 2 because he is directed to do so by the teacher's edict or the rule which he has learned. In short, according to the drill theory the teaching of arithmetic is best accomplished by analyzing the material to be taught into

$$\begin{array}{r}17\\28\\69\\\hline 114\end{array}$$

many separate elements, having the pupil memorize these elements whether he understands them or not, and then having him repeat or practice them (drill) until he can retain them permanently.

The drill theory is very popular. It probably is used more extensively in the grades of the elementary today than is either of the other two theories to be discussed. The reasons are obvious. In the first place, the drill theory makes life comparatively easy for the teacher. She assigns tasks for the pupils to do and, in stern task-master style, sees to it that these tasks are done. True, she may sugar-coat the dose by using games and drill devices and by stimulating competition, but the learning is largely rote learning and the tasks are easy to plan and simple to execute.

In the second place, many textbooks are constructed in such a manner as to promote the use of a method which is based upon the drill theory. New processes are illustrated, summarized in boxed-in, italicized rules, and then followed by pages of examples for practice. The books, like the teachers, show the pupils *how* to solve the examples and problems but give them little opportunity to think out and *discover* methods for themselves. However, marked improvement is seen in some recently published textbooks.

In the third place, the use of standardized tests has encouraged the use of methods based upon the drill theory. These tests set standards in terms of speed and accuracy of work. Consequently, teachers stimulated by a desire to make a good showing, drive hard on routine skills with little regard for understanding.

Finally, many books and articles devoted to the teach-

ing of arithmetic stress the analysis of the subject matter into unit skills and amount and distribution of practice but stress little the growth of number concept, the development of an understanding of our number system, and the rationalization of new processes.[4]

Faults of the drill theory. There is undoubtedly a place for drill in the learning of arithmetic and other school subjects as well as in the learning of many things out of school, but the drill theory, as a theory of arithmetic teaching, is seriously inadequate. In the first place, it sets for the child an impossible task. In addition, for example, there are 100 basic facts and 765 higher decade combinations whose sums are less than 100. It is quite out of the question to expect a pupil to *remember* these 865 sums and the many other things which must be known in order to add competently. In division, there are 90 basic facts, 324 facts with remainders for the divisors 2 to 9, only, 3960 examples having the one-digit divisors 2 to 9 and two-digit quotients, etc.[5] In the subtraction of fractions, Brueckner reports 53 types of examples. There are literally hundreds of types of problems. If arithmetic is taught as a mass of hundreds or thousands of separate and more or less unrelated items, the learning of arithmetic will not be merely difficult; it will be almost impossible. If arithmetic is to be effectively learned, full advantage must be taken of the unifying ideas and relationships which run through

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[4] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School,* Volume I, Primary Grades. New York: Silver Burdett Company, 1937. 410 pp. Chapters 1 and 3 and portions of other chapters develop these points.

[5] Morton, R. L., *op. cit.,* Chapter 9.

this mass of items. The prevalence of the drill theory is doubtless largely responsible for the fact that there are more failures recorded in arithmetic than in any other subject in the curriculum of the elementary school.

Secondly, methods based upon the drill theory rarely produce the results that they are supposed to produce. Chazal[6] found that pupils who had been drilled prematurely on the addition combinations *frequently* did not know the combinations and in spite of much drill used round-about procedures of their own for finding sums. Continued drill merely strengthened the inefficient habits which they had formed. Furthermore, *such drill did not develop in the pupils that ability to do* quantitative thinking which is the important objective of arithmetic instruction.

Finally, the drill theory is inadequate. The processes of arithmetic must be meaningful else drill will be ineffective. For example, a pupil by following directions multiplies the fraction $\frac{3}{4}$ by the fraction $\frac{2}{3}$. He "cancels" and multiplies numerators and denominators to obtain a new numerator and a new denominator. Under the most favorable conditions, he gets $\frac{1}{2}$ for his product. But when asked to explain or tell the meaning of what he has done, he is at a loss. He does not understand what he has done or why he did it. He may not even have an adequate understanding of what is meant by the fractions, $\frac{3}{4}$ and $\frac{2}{3}$, with which he has operated. And he knows no satisfactory way to satisfy himself that his answer, $\frac{1}{2}$, is probably right.

[6] Brownell, William A. and Chazal, Charlotte B. "The Effects of Premature Drill in Third-Grade Arithmetic." *Journal of Educational Research,* XXIX: 17-28, September, 1935.

Drill on an arithmetic process after the process has been developed and is understood is a valuable means of fixing the process and increasing the probability of its retention but a method of teaching which is based upon the drill theory frequently produces results which are disappointing. Such a method fails to reveal arithmetic as a system of related ideas, it strengthens inefficient habits which the pupils develop in spite of the teacher's hopes and intentions, and it fails utterly to develop the ability to do quantitative thinking.

The incidental-learning theory. The unsatisfactory results secured from a method founded upon the drill theory have led students of the teaching process to reject this theory and to formulate a new theory which, it was hoped, would better accomplish the purposes of arithmetic instruction. This theory, the incidental-learning theory, has gained ground rapidly since about 1920.

There are many varieties of the incidental-learning theory; in fact, it may be called a group of theories. These theories differ as to details but, in general, they provide that pupils will develop number concepts and acquire skill in the operations of arithmetic simply by engaging in lifelike activities which are only in part arithmetical in character. The activity is the main thing and arithmetic is learned quite incidentally. Advocates of this theory would postpone arithmetic until the third or the fourth grade. Indeed, some would postpone arithmetic instruction until the seventh or the eighth grade. Studies are reported now and then which seem to show that pupils who omit arithmetic for several

years are no worse off than are those who have systematic instruction in each of the grades.[7]

Faults of the incidental-learning theory. The first objection to the incidental-learning theory is that it does not produce satisfactory results. Opportunities to learn arithmetic through experiences in activities are dependent too much upon chance and are too desultory in character. The teaching of arithmetic requires definite planning and a systematic program if the essential elements are to be learned and learned well. Activities may be of great value as a means of motivating the study of arithmetic but there can be no assurance that the subject will be learned thoroughly if the learning is all incidental. Some things can be learned incidentally but a complicated subject such as division requires a carefully planned attack and systematized practice. The added interest provided by the activity may produce results as good as those secured through the use of a method which is based upon the drill theory but, as we have seen, such results are far from satisfactory. The incidental-learning theory should yield a much higher standard than the very mediocre one set by the drill theory if it is to be made the basis of instruction in arithmetic.

Hanna[8] and his associates found that the activity program provided a very small number of arithmetic experiences per week. A survey of the arithmetic re-

[7] Morton, R. L., *op. cit.*, pp. 40-42.

[8] Hanna, Paul R., *et al.* "Opportunities for the Use of Arithmetic in an Activity Program." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic*. New York: Bureau of Publications, Teachers College, Columbia University. 1935, pp. 85-120.

quired by the activities of a group of sixth-grade pupils, for example, reveals not only a very limited number of examples which these pupils had to solve but also a wide scattering of these examples over many phases of arithmetic with not enough on any one phase to assure effective learning. Hanna observes that the activities selected are largely a matter of chance and that there is no assurance that they will introduce the pupils to the important elements of the school curriculum. "Functional experiences of childhood are alone not adequate to develop arithmetic skills."[9]

The incidental-learning theory is based upon the assumption that children will not learn arithmetic unless they are interested in it and that interest must be secured through the medium of activities. No one will deny the role of interest in learning. The drill theory did much to destroy interest and children undoubtedly find an absorbing interest in well-selected activities. But we are led here to a second objection to the incidental-learning theory. *The pupil's interest lies in the activity rather than in the arithmetic.* Such interest as he has in arithmetic is secondary. It is superficial. He is interested in the completion of the activity. Consequently he uses a little arithmetic only as a means of accomplishing that end. If the activity left him with an absorbing interest in arithmetic for its own sake, a most worth while purpose would be accomplished. The activity may help; but a teacher cannot depend entirely upon activities as a means of developing an interest in arithmetic. There is always the danger that when the

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[9] *Ibid.*, p. 118.

pupil *has completed an activity, he will simply be ready* for another activity.

Good teaching leaves pupils interested in arithmetic for its own sake. Some children like arithmetic and want to learn more of it. Many others will learn to like it if they have teachers who understand both arithmetic and children and who are able to help them to discover the nature of the number system, the meanings of the operations, and the reasons for what they do, as well as the usefulness of the subject. Under the incidental-learning theory, arithmetic is likely to be a mere tool, just a means to an end. It is far more than that; it is a method of thinking. Children who really learn arithmetic get more than a superficial acquaintance with the fundamental operations. They learn to do quantitative thinking. If arithmetic is to be learned, it must be attacked directly, not merely indirectly through the medium of an activity.

There is more of a logical system in arithmetic than in any other elementary school subject. One topic leads to another. Consequently they must be studied in proper sequence. Pupils should not undertake to multiply fractions before they have become proficient in multiplying integers. They should not undertake percentage before they have developed an acquaintance with common and decimal fractions. They should not attempt to add columns of numbers before learning the fundamental combinations and higher-decade addition. If the learning of arithmetic is to be incidental, there can hardly be proper attention to this matter of sequence and organization. Learning becomes haphazard and superficial. As Brownell expresses it, "Arithmetic can

hardly be learned as it should be learned: relationships, dependencies, mathematical principles may easily escape the notice of the teacher, and so of the learner."[10]

Finally, the activity method of learning arithmetic is slow and time consuming. The activities may be worth while for their own sake or for purposes other than the learning of arithmetic and they may help in building up a desire to learn arithmetic, but eventually there must be a direct and systematic attack upon arithmetic as a subject of study if it is to be learned by the pupils.

The "meaning" theory. As Brownell remarks, this theory of arithmetic is not easily named.[11] It has points of similarity with the other two theories which have been discussed but it is different from either of them. It recognizes the value of both drill and activities but it holds that arithmetic is an integrated system of ideas, principles, and processes rather than a miscellaneous assortment of unrelated elements and that it can not be learned satisfactorily through incidental teaching. The meaning theory demands that pupils shall understand what they "learn," and that they shall get the ideas and principles which make arithmetic an integrated system, as well as skill in the processes. Because of the emphasis which this theory places upon meaning, we have followed Brownell's lead in calling it the meaning theory.

Volume I of this series[12] is permeated with the meaning theory of arithmetic although the different theories

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[10] Brownell, W. A., *op. cit.*, p. 17.

[11] Brownell, W. A., *op. cit.*, p. 19.

[12] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company. 1937. 410 pp.

are not separately discussed in that volume. In brief, it is held that children should *understand* numbers before they learn to operate with them, that the *meaning* of the operations should be made clear before these operations are learned, that new steps and processes should be *discovered* by the children from their relationships to steps and processes already learned, that *drill does not make arithmetic meaningful* but it is of value in fixing the skills and maintaining them at a high level of usefulness,[18] and that the *ideas and principles* of arithmetic rather than skill in computation are the means of developing in children the ability to do quantitative thinking.

A method based upon the meaning theory is constructed so as to assist children in *understanding* arithmetic. There are various ways in which this may be done. In the first place, it is recognized in the meaning theory that arithmetic is abstract. The fact that arithmetic is abstract makes it difficult. If children are really to learn arithmetic, they must develop the ability to do abstract thinking in quantitative situations. Thus, if a child sees four apples, or four pencils, or four circles on the blackboard, he is having concrete or semiconcrete experiences involving the idea of four. But the idea of four, or "four-ness," is not concrete but abstract. It is a concept which has been developed through the intelligence of the learner and which the mind of the learner eventually imposes upon the concrete objects or experiences. A method based upon the meaning theory provides that the child shall move

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[18] Cf. Stretch, Lorena B. *"The Value and Limitations of Drill in Arithmetic," Childhood Education,* XI: 413-416, June, 1935.

slowly and gradually from the concrete stage of number experience through the intermediate stages where pictures are used and where numbers are represented by such semi-concrete means as circles or dots or lines, to the final stage where numbers are represented by abstract symbols.[14]

Secondly, the meaning theory provides that the rate at which the pupils proceed shall depend upon the difficulty of the material to be learned. It seems so obvious that rate of progress should depend upon difficulty that the point need not be mentioned. But the drill theory, which gives scant attention to understanding, naturally provides that pupils go at the same rate over material which they are unlikely to understand as over material which they may understand. And the incidental-learning theory, by bringing arithmetic topics into activities in a hit-or-miss, haphazard manner, according to the needs of the moment, does not provide ample opportunity for comprehension when the topics are unfamiliar and difficult.

A systematic attack upon a new topic, such as division, if planned in accordance with the meaning theory, provides that the work shall be organized into a carefully graded series of steps and that the pupils shall understand each step and acquire a fair degree of competence in it before the next step is undertaken. Activities and experiences which require division are found and used and drill on the necessary techniques is provided, but the practice which pupils receive is not limited to that required in the activities, and drill is not provided prematurely.

[14] Morton, R. L., *op. cit.*, pp. 3-5 and 12-18.

According to the meaning theory, pupils go slowly in the early stages of a new and unfamiliar topic. Furthermore, so far as is consistent with the requirements of logical organization, the easier phases of a topic are learned before (sometimes long before) the harder phases. Thus, some phases of the subject of common fractions may be considered in the primary grades,[15] even as early as the first grade,[16] while the more difficult phases will not be undertaken before the pupils reach the fifth or the sixth grade. Schools in which instruction is planned according to the drill theory usually have all of the work in fractions confined to one or two grades, chiefly the fifth. Schools following the incidental-learning theory may have the various topics in fractions in any order and at any time.

This spreading of topics is gradually becoming a more and more common practice in the arithmetic of the elementary school. It is well illustrated by the work in denominate numbers. Whereas, the older school, basing its plan of organization chiefly upon the drill theory, concentrated this subject largely in a single grade, the current tendency is to spread the various phases of the subject over all of the grades and even to consider the parts of a single "table" at widely separated times, according to difficulty and the pupils' needs. This is done simply because the subject becomes more meaningful to pupils when so treated. Thus we see that the meaning theory encourages the spreading of a topic like fractions so that the elements may be taken

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[15] Morton, R. L., *op. cit.*, pp. 32, 282-285, 335-341, 386-387, 394-395.

[16] Polkinghorne, Ada R. "Young Children and Fractions." *Childhood Education*, XI: 354-358, May, 1935.

up according to their difficulty, as well as according to their usefulness, and a topic like measurements so that usefulness and interest may be given proper considera-tion. At the same time, there is no violation of an essential logic of organization.

Finally, the meaning theory emphasizes the relationships which exist between topics and parts of topics. Higher decade addition is seen as simply an extension of the fundamental combinations, that is $48 + 4$ is an extension of $8 + 4$. Percentage is not presented as a new idea but as another way to express that with which the pupils are already familiar. Percentage is seen to be intimately related to both decimal fractions and common fractions. At all times, an effort is made to teach arithmetic as a body of ideas, principles, and relationships. Taking full advantage of these principles and relationships means economy of time and energy and a much better understanding of the subject.

The meaning theory and quantitative thinking. The prime objective of arithmetic teaching, according to the meaning theory, is the development in the pupils of the ability to do quantitative thinking. Much more is expected of the pupils than the mere ability to compute. They are expected to be able, eventually, to analyze a situation, to isolate the essential elements for further attention, to make decisions, and to answer pertinent questions. The ability to do quantitative thinking does not develop rapidly; it comes slowly, much more slowly than many teachers seem to realize. It requires frequent stimulation, guidance, and checking by an expert teacher. Such an ability will rarely be developed by a pupil if he is left to his own devices as is too often

done when a pupil is taught according to the methods of either the drill theory or the incidental-learning theory. It is the teacher's responsibility to see that the pupils grow in the ability to think quantitatively. Brownell says,

"In meeting this responsibility the teacher is unwise who measures progress purely in terms of the rate and accuracy with which the child secures his answers. These are measures of efficiency alone, not of growth. It is possible for the child to furnish correct answers quickly, but to do so by undesirable processes. The true measure of status and development is therefore to be found in the level of the thought process employed. If the teacher is to check, control, and direct growth, she must do so in terms of the child's methods of thinking. If the child tends to rest content with a type of process which is low in the scale of meaning, she will lead him to discover and adopt more mature processes. If she asks him to "explain" an exercise written on the blackboard or on his paper, she will not be satisfied merely to have him read what he has written; she will insist upon an interpretation and upon a defense of his solution. She will make the question, "Why did you do that?" her commonest one in the arithmetic period. Exposed repeatedly to this searching question, the child will come soon to appreciate arithmetic as a mode of precise thinking which derives its rules from the principles of the number system."[17]

What theory shall we accept? Desirable outcomes in arithmetic have seldom been attained through the use of the drill theory. A few persons taught according to

[17] Brownell, W. A., *op. cit.*, pp. 28-29.

the drill theory, have learned arithmetic well but this has been due, apparently, to superior intelligence or outside influence. At any rate, those who have learned well in schools where the methods of the drill theory predominate are far outnumbered by those who have not learned well. They have learned arithmetic in spite of the drill theory rather than because of it.

The drill theory is still widely accepted but it has given way in many places to the theory of incidental learning. The incidental-learning theory has gained rapidly in recent years and is now very much in vogue in many so-called progressive schools, although there are already signs that it is being modified so as to provide more opportunity for drill and so that more attention can be given to meaning. There are reasons for grave doubts as to the efficacy of a method which is based largely upon the theory of incidental learning.

The meaning theory holds much greater promise than does either of the other two. It recognizes merit in the incidental-learning theory and some merit in the drill theory. These merits it incorporates but it is not a compromise between these two theories. It is a different theory with added features and advantages. It holds that arithmetic is a system of interrelated principles, that children will become interested in learning it for its own sake as well as for its obvious usefulness, that understanding must come before drill,[18] that new steps must be discovered by pupils rather than merely accepted by them, that drill if not given prematurely has an important function to perform, and that through arithmetic children should learn to do quantitative

[18] Morton, R. L., *op. cit.*, pp. 6-8.

thinking. Brownell says, "The basic tenet in the proposed instructional reorganization is to make arithmetic less a challenge to the pupil's memory and more a challenge to his intelligence."[19]

Functions of arithmetic instruction. Reference was made on page 3 to the four functions of arithmetic instruction which were suggested by the Reviewing Committee of the *Twenty-Ninth Yearbook* of the National Society for the Study of Education.[20] They are:

1. The computational function
2. The informational function
3. The sociological function
4. The psychological function

These four functions are outlined again by Brueckner in the *Thirty-Fourth Yearbook* of the National Society.[21]

These four functions of arithmetic do not always exist in the sharply separate fashion which their enumeration may suggest but are closely interrelated. In many a life situation requiring arithmetic, two, three, or all four of these functions find expression. For example, if one is to think clearly and intelligently about the financial problems incident to social security, he must understand the economic conditions and the movements for social welfare which have brought about social security legislation, he must be able to under-

[19] Brownell, W. A., *op. cit.*, p. 31.

[20] This Committee consisted of Leo J. Brueckner, Chairman, J. C. Brown, J. R. Clark, H. L. Harap, Ernest Horn, C. H. Judd, L. A. King, Worth McClure, R. L. Morton, Elma A. Neal, W. J. Osborn, J. R. Overman, F. G. Pickell, and G. M. Ruch.

[21] Brueckner, Leo J. "Diagnosis in Arithmetic." National Society for the Study of Education, *Thirty-Fourth Yearbook*. Bloomington, Illinois: Public School Publishing Company, 1935, pp. 269-272.

stand the computations involved in determining the contributions to be made by employers and by employees and the accumulation of these amounts by compound interest, he must appreciate the advantages and the possible disadvantages of plans for social security, he must see the relation of such a program to his own present and future welfare, and otherwise and in many details he must be able to think in an orderly manner through the social, economic, and quantitative aspects of the subject. It will be noted that only a portion of the situations involving arithmetic require computational arithmetic.

The computational function. The typical school of the past has made computation the decidedly major objective of arithmetic instruction. Even at the present time, there is probably an overemphasis upon computation in the majority of the schools. Steel[22] found that in a group of schools which he studied, more than five-sixths of the time devoted to arithmetic went to the development of computation processes and drill upon them. Brueckner[23] secured reports of observations of 505 arithmetic lessons in schools widely scattered over the United States and found that the major emphasis

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[22] Steel, H. J. "Time Activity Analysis Technique Applied to the Supervision of Arithmetic." *The Second Yearbook of the National Conference of Supervisors and Directors of Instruction.* New York. Bureau of Publications, Teachers College, Columbia University, 1929, pp. 133-144.

[23] Breuckner, Leo J. "An Analysis of Instructional Practices in Typical Classes in Schools of the United States." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 32-50.

was placed upon the solution of examples and problems and that little attention was given to the social applications of number or to the development of an interest in number.

There is a growing conviction to the effect that teachers should be more concerned about the other functions of arithmetic. This does not mean that pupils become more skilful at computation than they should nor does it mean that more time should be devoted to the subject of arithmetic. In fact, complaints are common to the effect that the computational results are not what they should be and that arithmetic now consumes too large a portion of the day's program. Naturally, if more attention is to be given to functions other than the computational and the total amount of time devoted to arithmetic is not to be increased, less time must be spent directly on the development of skill in computation. How, then, can we expect to avoid still less satisfactory results in computation?

The answer to this question lies in the fact that if more attention is given to the informational, the sociological, and the psychological functions, arithmetic will be far more meaningful to the pupils, they will be more greatly interested in it, and they will do better in computation as well as in other phases. Furthermore, if methods of teaching are based upon the meaning theory rather than upon the drill theory, which seems to be the more common practice where the computational function is stressed so greatly, or the theory of incidental learning, we can expect greater progress per unit of time. Much of the prevailing practice in computation is carried on in a routine, wooden manner and is so

uninteresting to the pupils that satisfactory results cannot be expected, even in computation.

The suggestion is sometimes offered that the common use of calculating machines makes skill in computation less important than it used to be. It is held that the ordinary individual has little use for computational skills because most of his figuring is done for him by some one else who always has a machine at hand. To the author, this conclusion seems to be unsound. Our modern complex society supplies far more situations which involve computation than did the society of preceding generations. To be sure, most of the computing *can* be left to the expert but there is always the possibility, amounting sometimes to a probability, that the individual will not only have his computing done for him by the expert but also that he himself will be "done" by the expert. Instances in which customers receive the wrong change after making a purchase are more common than most persons realize. Few persons know how costly installment buying is or how high the charges are which are assessed by concerns making a specialty of lending small sums of money. The advantage or disadvantage of purchasing large or small quantities of various commodities is seldom realized. In many ways, the individual who can compute has a decided advantage over the one who leaves his computing to others. Since a very small proportion of the population yet has access to computing machines, the ordinary individual must do his computing by the methods of elementary school arithmetic or not at all. Instead of having less need for skill in computation, we have far greater need than was the case with earlier

generations. We should set higher computational goals in school rather than lower. In our zeal for understanding, we should not neglect the skills. Skill in the fundamental operations is worth while; and *skill with understanding is education.*

The informational function. The informational function is concerned with the enrichment of the arithmetic course by enlarging upon topics in such a way as to make them vital and real and to give them greater meaning. In the study of measures, for example, the computational function merely provides that after the pupil has memorized the necessary facts from the tables, he will solve examples which use these facts, examples requiring reduction, addition, subtraction, and the like. The informational function will also provide that the pupil learn these measures better through use, that he gain some knowledge of the origin of measures, and that he learn something about the status of comparable measures in other countries if differences exist. For example, information on the historical development of the gallon and of the difference between the gallon used in this country and that used in Canada may be interesting and worthwhile.

There are many opportunities in the study of arithmetic to acquire information which will increase the pupil's understanding of arithmetic facts and processes, add greatly to his interest in the study of this subject, and indicate the many intimate contacts between the arithmetic studied in school and the complicated and ever-varied lives which children and adults lead. This information will have to do with money, banking, salaries and wages, taxation, investments, insurance and

other social institutions; with measures and measurements and their dependence upon number; with the instruments which have been developed as a means of making precise measurements; with the historical development of number, measures, and various number applications; and with the knowledge necessary to a more intelligent understanding of the production, distribution, and consumption of goods."[24] So far as the arithmetic of the intermediate grades is concerned, many of the informational aspects of topics will be treated in a very elementary way, more serious treatment being reserved for later grades, but to neglect informational arithmetic in our zeal for the computational and other functions is to fail to take advantage of many attractive opportunities to make arithmetic interesting, meaningful and more worthwhile.

Buckingham suggests that the number facts themselves are informational.[25] "They are," he says, "the informational basis of computation." Likewise, many other elements of the arithmetic curriculum not only form the frame work for the computational function but have significant informational aspects also. Schools in which arithmetic instruction consists very largely of computation are prone to present these elements as mere abstractions, neglecting the informational content. If arithmetic topics are developed in a

[24] Brueckner, Leo J. "Diagnosis in Arithmetic" National Society for the Study of Education, *Thirty-Fourth Yearbook*. Bloomington, Illinois: Public School Publishing Company, 1935, p. 270.

[25] Buckingham, B. R. "Informational Arithmetic." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic*. New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 51-73.

meaningful way, they will not only contribute to the development of computational skills but also will furnish information which will put meat on the otherwise dry bones. If time is spent in developing adequate number concepts before the number facts are treated as abstractions, pupils can be expected to gain this richer meaning and better understanding.[26]

The sociological function. Arithmetic finds many applications in practical affairs. It became a school subject of study primarily because the knowledge and the skills which it engenders were very definitely needed. We have seen (page 1) that the subject soon became highly formalized; the processes were not made meaningful to the pupils nor were the applications to social affairs made apparent. In many schools, the subject continues in this state. The drill theory reigns supreme and as a result emphasis is given largely to the computational function.

If more emphasis is placed upon the sociological function—the uses of arithmetic in production, distribution, consumption, business and government affairs, and social relationships—the subject will take on new interest, it will be learned more easily and more successfully, and it will be used after school days are over to solve a greater number of problems.

In teaching the topics of arithmetic, an attempt should be made to find the uses of these topics in the affairs of individuals and groups. A study of the experiences which people have with special reference to their quantitative phases readily reveals the need for arithmetic. It also reveals, readily enough, the fact that

[26] Morton, R. L., *op. cit.,* Chapter 1.

more arithmetic might be used in these affairs if the persons concerned knew more of arithmetic.

A few studies of the business and social uses of arithmetic have led to the conclusion that very little arithmetic is needed. Wilson found in a survey of the uses of arithmetic that many topics and parts of other topics could be dispensed with since instances of their occurrence were not found.[27] It was recommended, for example, that division of fractions be almost completely eliminated. In the study of fractions, no attention would be given to sevenths or ninths and very little to sixths. Bowden came to the conclusion that the consumer has little use for arithmetic beyond the simplest fundamentals.[28] Such studies as these make a real contribution in that they point the way toward the elimination from the course of study of obsolete topics and unreal problems and examples. But to confine instruction to those topics which one discovers to be in use when he makes a survey is a short-sighted policy. In the first place, no survey will reveal adequately the uses of arithmetic. Limitations of time and money make it impossible to contact a sufficient number of persons and to remain with these persons long enough to learn what arithmetic they really use. Secondly, as already indicated, there is excellent reason to believe that most persons would use more arithmetic if they knew more arithmetic. Casual observation soon reveals that of two persons having similar experiences, the one who knows

[27] Wilson, Guy Mitchell. *What Arithmetic Shall We Teach?* Boston: Houghton Mifflin Company, 1926, 149 pp.

[28] Bowden, A. O. *Consumers' Uses of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1929, 61 pp.

the more arithmetic finds the greater number of uses for arithmetic. Finally, if arithmetic is a body of interrelated principles and ideas, it cannot be adequately presented as a mass of fragments which happened to be found when a survey of use was made. Some attention to phases of a topic which are seldom used is necessary if we would present a unified and well rounded picture of the subject.[29]

Buswell says that the social values are the most important values of arithmetic and that a survey of computational practices is not sufficient for determining what these values are. He suggests that there are two major types of material in arithmetic. The first of these is the number system which must be learned as a system. As one learns the interrelations of this system in addition, subtraction, multiplication, and division, he must acquire ability in computation. The second is the social application of number. It must come eventually to be a part of the thinking habits of those who study arithmetic.[30]

If an individual is to be an intelligent consumer, is to make desirable selections of food, clothing, shelter, labor-saving devices, and educational and recreational facilities, he must continually give his attention to matters of a quantitative character. Harap has shown

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[29] Criticisms of Bowden's Study are contained in articles in *The Mathematics Teacher*, Vol. XXIII, by C. N. Shuster (pages 180-184) and William E. Roth (pages 467-473) March, 1930, and December, 1930, respectively.

[30] Buswell, G. T. "The Relation of Social Arithmetic to Computational Arithmetic." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic*. New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 74-84.

that the consumer must have quantitative information of many kinds and must be able to use a wide variety of arithmetic processes in many and varied situations.[31] He lists instances requiring calculation in (1) food measurement, (2) household measurement, (3) fuel measurement, and (4) clothing measurement. Food measurement requires the consumer to use his arithmetic in such matters as finding the loss in buying sugar in small cartons; finding the difference in price between fresh and canned meats; finding differences in costs between public markets and retail stores; finding the amount saved by doing home cooking; finding the cost of fractional contents of a bag, box, etc.; determining the amount which can be purchased with a dime, a quarter, or other monetary units; increasing or decreasing the food budget to correspond to increases or decreases in income; finding the amount saved by making quantity purchases; checking bills; reading tables; reading recipes; and many others. The reader will readily see how household measurement, fuel measurement, and clothing measurement also require the consumer to use his arithmetic.

Judd tells a story to the effect that during the World War the War Department ordered a quantity of blackberry jam which was 40 times as great as the quantity which had been used in peace times simply because the number of soldiers was then 40 times as great as it had been before the War. A clerk in the War Department had no difficulty in making out the order for he was quite competent in computational arithmetic but he

[31] Harap, Henry, *The Education of the Consumer*. New York: The Macmillan Company, 1924, pp. 320-324.

did not even guess that it was impossible to fill so large an order for blackberry jam. There were not enough blackberries. The clerk was not touched by the informational or the sociological aspects of the problem.[32]

The psychological function. We have referred to the importance of developing in pupils the ability to do quantitative thinking. One can not think quantitatively beyond the elementary level of making simple comparisons, unless he has been trained in number. Thus, number becomes a method of thinking.

Judd has repeatedly called mathematics, including elementary arithmetic, a mode of thought. He stresses the importance of general ideas as opposed to specific skills and isolated bits of information. To quote,

"General ideas of what a group is and of the way in which groups can be rearranged come only through contact with many different groups. When once achieved, the higher type of thinking is continually enlarged and refined by being employed in different connections. The idea of addition, for example, is not a fixed notion. It cannot be given to a pupil. It must be acquired through a long period of enlargement of experience, in the course of which various kinds and degrees of recombinations of groups are experienced.

"A very false and misleading psychological doctrine has been current in some quarters. It is said that, in the pupil's mind, addition is nothing but a long list of particular combinations. Such psychology does not provide any explanation of the general or abstract idea of addition as distinguished from particular experiences.

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[32] Judd, Charles H. "Informational Mathematics *versus* Computational Mathematics." *Mathematics Teacher*, XXII: 187-196, April, 1929.

When a teacher thinks of addition merely as a series of separate processes, he makes no provision in his teaching scheme for the operation of the generalized idea. When, on the other hand, a teacher thinks of the idea of addition as a generalized idea, he will strive to emphasize this idea. He will have confidence that, if the pupil is taught to acquire an insight into the general idea, the pupil will, because of the inner drive of this general idea, help to train himself in particular cases not explicitly included in the lessons taught in school. When once acquired, a general idea is a dynamic fact in experience."[88]

This interpretation of the function of arithmetic teaching promptly leads one to conclude that a course in arithmetic cannot be properly organized solely in terms of narrow utilitarian aims. Arithmetic serves utilitarian purposes, to be sure; but it does more than this. If properly taught, it serves to give pupils an intellectual equipment without which effective thinking in situations which are quantitative in character cannot be done. Quoting again from Judd,

"There is a tendency in many quarters to think of arithmetic as one of the subjects which can be drastically reduced. It is sometimes urged that arithmetic be cut down to the point where only its practical applications will be included in the curriculum. The conclusions to which the study reported in this monograph lead are diametrically opposed to the doctrine that

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[88] Judd, Charles Hubbard. *Psychological Analysis of the Fundamentals of Arithmetic.* Chicago: The University of Chicago, 1927, pp. 109-110.

arithmetic should be reduced to a few exercises in practical calculation. If the studies which have been reported prove anything, they prove that the general ideas which are developed through contact with numbers can be cultivated in the individual only through a broad acquaintance with the properties of a highly perfected number system. To eliminate number instruction from the schools or to give it only a minor place would be to suppress one of the most significant general ideas that the race has evolved. To reduce arithmetic to a few practical applications would be to neglect the general idea of precise thinking on which our mechanical and scientific civilization rests."[34]

"Curriculum makers should be urged to recognize the fact that the curriculum is made for the purpose of training minds, not for the purpose of reflecting the immediate needs of practical living. There are psychological requirements which must be met in curriculum-making, and these are infinitely more important for the future of the race than are the practical adjustments of trade and industry. When the Arabic numerals came into Europe, they found many skills highly developed, but they found little exact science. In the short span of time which is marked by the lapse of less than four centuries, the industrial revolution has substituted machines, which are products of precision in thinking, for human hands, and exact science for blind trial. The achievements of the industrial revolution would have been utterly impossible without number. It is no trivial

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[34] Judd, C. H., *op. cit.*, p. 116.

assault on the intellectual institutions of the race that is made by those who would dispense with mathematical education or reduce it to a bare minimum."[35]

"All the experiments and analyses reported in this monograph lead to the conclusion that general ideas are the most important products of instruction in arithmetic. The fundamentals of arithmetic are general ideas and general formulas, not a multitude of special skills."[36]

The psychological function includes many aspects of arithmetic ability which may not be given consideration in a school where most of the emphasis is placed on the computational function. These include, according to Brueckner, an understanding of the structure of the number system, the development of quantitative concepts, the ability to use quantitative methods as a basis for accurate, orderly thinking, the ability to discover relationships, the appreciation of geometric design, the habit of using quantitative techniques in studying various problems and issues, and others.[37] Obviously, some of these will not be well developed in the intermediate grades but the engendering of such habits and abilities certainly may be begun in an elementary way in the intermediate grades.

Why should children study arithmetic? It hardly seems that the value of arithmetic as a part of the curriculum of the elementary school can be questioned. Arithmetic consisting largely of mere computation and taught by a method based upon the drill theory is of

[35] *Ibid.*, pp. 116-117.
[36] *Ibid.*, p. 117.
[37] Brueckner, *op. cit.*, p. 271.

doubtful value when judged by results secured in schools where such practices prevail. But arithmetic which gives due consideration to the informational, the sociological, and the psychological functions, as well as to the computational function, is an indispensable part of the training of the young.

QUESTIONS AND REVIEW EXERCISES

1. Think back over the kind of arithmetic which you had when a pupil in the primary grades. What theory of arithmetic instruction predominated? Was the study of arithmetic interesting to you? To other members of the class? Was it meaningful?

2. Are there pupils in the schools who are interested in arithmetic regardless of the prevailing theory of instruction? If so, what kind of pupils are they? Are they numerous? Would they enjoy arithmetic more and get more out of it if it were taught according to the meaning theory?

3. Have you known persons who questioned the value of arithmetic as a subject of study? What is the basis of such a point of view?

4. What is the drill theory of arithmetic instruction? Characterize briefly the procedure of a teacher who uses a method which is based upon the drill theory.

5. What reasons can you give for the obvious popularity of the drill theory?

6. Is there a place for drill in arithmetic? If so, when should it come? What are the effects upon pupils of premature drill?

7. What is the theory of incidental learning? Characterize briefly the procedure of a teacher who uses a method which is based upon the incidental-learning theory.

8. What benefits arise from the use of projects or activities in elementary education? How can a teacher guarantee

these benefits without sacrificing the large gains which should come from the study of arithmetic?

9. What are the faults of the incidental-learning theory as applied to the teaching of arithmetic?

10. Suggest some out-of-school activities which require the use of drill. Some which are apparently accomplished largely through the method of incidental learning?

11. Describe activities which you have seen carried on in school and which made important contributions to the learning of arithmetic. Have you known activities to be used in such a way as to neglect opportunities for the learning of arithmetic?

12. Can we expect pupils to be interested in learning arithmetic for its own sake? How may activities keep pupils from developing such an interest in arithmetic?

13. What is the meaning theory? Does it recognize merit in drill? In activities? Is it a compromise between the drill theory and the incidental-learning theory?

14. How may the incidental-learning theory bring a pupil to the consideration of a topic in arithmetic for which he is not yet prepared. Is such premature use of an arithmetic topic likely to result in meaningful learning? If not, is the result secured likely to be better than that secured through the use of the drill theory?

15. Why is it necessary to make a planned, systematic attack upon many topics in arithmetic?

16. What topics in arithmetic are being spread over a longer period of time than was formerly the case? What are the advantages in such spreading?

17. What is meant by the ability to do quantitative thinking? How does the meaning theory foster the development of such thinking?

18. Summarize the advantages of the meaning theory over the drill theory. Over the incidental-learning theory.

19. Incidental learning is sometimes facetiously referred to as "accidental" learning. Do you believe that the learning of arithmetic under the incidental-learning theory is ever accidental?

20. How do you account for the emphasis which has been placed upon computational arithmetic. Examine the textbooks in use in your community. Do they give adequate attention to functions other than the computational?

21. If more attention is given to arithmetic functions other than the computational, will computational skills decline?

22. Does the growing use of calculating machines justify a lower level of achievement in the fundamental skills of arithmetic?

23. To what extent should the average citizen leave his calculating to the expert? Since it may be assumed that the results secured by the expert are correct, what, if any, loss would the average citizen suffer if he left his calculating to the expert?

24. What opportunities for the exercise of the informational function do you see in the arithmetic of the intermediate grades?

25. What, in general, is the nature of the results secured by those who make surveys of the uses of arithmetic in the ordinary individual's affairs? Why can not the arithmetic curriculum be dictated by these results?

26. Suppose it is found that the average individual uses only 60 per cent of the arithmetic which he learned in school. Does this mean that he had no use for the remaining 40 per cent, or that he had forgotten it, or both? If it means that he had forgotten it, and the arithmetic curriculum is then correspondingly reduced, would still less be used by the next generation? What would be the eventual result of this process?

27. What does Judd mean by "general ideas" as important outcomes of the study of arithmetic?

28. Look over the list of abilities which Brueckner lists as belonging under the psychological function. Describe a concrete situation for each.

29. Which of the four functions of arithmetic do you think is most seriously neglected in schools with which you are acquainted?

30. What is the answer to the question, "Why should children study arithmetic?"

CHAPTER TEST

Read each statement and decide whether it is true or false. A key for this test will be found on page 533.

1. Warren Colburn recommended that arithmetic teaching be more formal.
2. Colburn's arithmetics effected an increase in the amount of time devoted to arithmetic.
3. Standard tests in arithmetic tended to increase the emphasis placed upon drill.
4. The three theories of arithmetic instruction include one called the sociological theory.
5. According to the drill theory, the learning of arithmetic consists primarily in the acquisition of a large number of specific facts and skills.
6. The drill theory places emphasis upon having the pupils discover new facts and processes for themselves.
7. The drill theory is still a popular theory of arithmetic instruction.
8. There is a place for drill in the learning of arithmetic.
9. Results secured through the use of the drill theory are usually unsatisfactory.
10. According to the drill theory, the pupils will always see reasons for what they learn to do.

11. Arithmetic is a system of interrelated ideas.

12. There are many varieties of the incidental-learning theory.

13. According to the drill theory, arithmetic is learned through the use of projects or activities.

14. Studies of the activity program prove that it is a satisfactory means of learning arithmetic.

15. The incidental-learning theory is based upon the assumption that interest in arithmetic must be secured through the medium of activities.

16. Pupils can not be expected to be interested in arithmetic for its own sake.

17. The incidental-learning theory gives much attention to the organization and sequence of topics in arithmetic.

18. The best way to learn to do quantitative thinking is to acquire skill in computation.

19. In arithmetic, children must learn to do abstract thinking.

20. The current tendency is toward a plan which requires that all of measurement be taught in a single grade.

21. The meaning theory holds much greater promise than does either of the other two.

22. Life situations requiring arithmetic frequently involve more than one of the four functions of arithmetic instruction.

23. The most conspicuous of the four functions in the past has been the informational function.

24. If greater emphasis is to be placed upon functions heretofore neglected, more time must be devoted to arithmetic.

25. Calculating machines are making computation much less important to the ordinary individual.

26. It was suggested that the ordinary individual should have his computing done for him by the expert.

27. Buckingham says that the number facts are the informational basis of computation.

28. Surveys of the social uses of arithmetic lead to the conclusion that not enough arithmetic is taught in the schools.

29. Judd says that the fundamentals of arithmetic are general ideas and general formulas.

30. All four of the functions of arithmetic should be stressed in a well-balanced program.

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2. Brownell, William A. "Psychological Considerations in the Learning and Teaching of Arithmetic." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 1-31. This very excellent article deals with the three theories of arithmetic instruction.

3. Brownell, William A. and Chazal, Charlotte B. "The Effects of Premature Drill in Third-Grade Arithmetic." *Journal of Educational Research,* XXIX: 17-28, September, 1935. Shows that if drill comes before understanding, the results are very unsatisfactory.

4. Brueckner, Leo J. "Diagnosis in Arithmetic." National Society for the Study of Education, *Thirty-Fourth Yearbook.* Bloomington, Illinois: Public School Publishing Company, 1935, pp. 269-302. The four functions of arithmetic

instruction are outlined in the first few pages of the article.

5. Brueckner, Leo J. "An Analysis of Instructional Practices in Typical Classes in Schools of the United States." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 32-50. Reveals the types of instruction found in a wide selection of arithmetic classes. Much stress was placed on the computational function.

6. Buckingham, B. R. "Informational Arithmetic." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 51-73. Stresses the importance of enriching arithmetic instruction by supplying more informational material.

7. Buckingham, B. R. "Significance, Meaning, Insight—These Three." *The Mathematics Teacher,* XXXI: 24-30, January, 1938. This excellent article presents a discussion of three important phases of arithmetic teaching.

8. Buswell, G. T. "The Relation of Social Arithmetic to Computational Arithmetic." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 74-84. Emphasizes the fact that a desirable arithmetic program contains much material of a social character.

9. Hanna, Paul R. *et al.* "Opportunities for the Use of Arithmetic in an Activity Program." National Council of Teachers of Mathematics, Tenth Yearbook, *The Teaching of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1935, pp. 85-120. Shows that an activity program provides insufficient opportunity for the learning of arithmetic.

10. Harap, Henry. *The Education of the Consumer.* New

York: The Macmillan Company, 1924, pp. 320-324. Provides a list of arithmetic uses found in food measurement, household measurement, fuel measurement, and clothing measurement.

11. Judd, Charles H. "Informational Mathematics versus Computational Mathematics." *Mathematics Teacher*, XXII: 187-196, April, 1929. Indicates that the arithmetic curriculum should be enriched by the inclusion of more informational material. Expresses a critical reaction toward Wilson's *What Arithmetic Shall We Teach?*

12. Judd, Charles Hubbard. *Psychological Analysis of the Fundamentals of Arithmetic*. Chicago: The University of Chicago, 1927, 121 pp. The entire monograph is well worth reading. The importance of general ideas is stressed in several passages.

13. Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades*. New York: Silver Burdett Company, 1937, 410 pp. The necessity of developing number concept and of making new processes meaningful is emphasized in Chapters 1 and 3 and elsewhere throughout the book.

14. National Society for the Study of Education, *Twenty-Ninth Yearbook*. Bloomington, Illinois: Public School Publishing Company, 1930, 709 pp. An excellent collection of articles dealing with various phases of arithmetic. The Reviewing Committee's critique, pp. 679-709, suggests that the computational function is overemphasized.

15. Polkinghorne, Ada R. "Young Children and Fractions." *Childhood Education*, XI: 354-358, May, 1935. Shows that children in the primary grades have picked up considerable information about fractions. Suggests the importance of spreading this topic.

16. Steel, H. J. "Time Activity Analysis Technique Applied to the Supervision of Arithmetic." *The Second Year-*

book of the National Conference of Supervisors and Directors of Instruction. New York: Bureau of Publications, Teachers College, Columbia University, 1929, pp. 133-144. Reveals that most of the arithmetic period is spent on drill activities.

17. Stretch, Lorena B. "The Value and Limitations of Drill in Arithmetic." Childhood Education, XI: 413-416, June, 1935. Indicates that drill has a place if given at the right time and in the right manner.

18. Thiele, C. L. "An Incidental or an Organized Program of Number Teaching?" *Mathematics Teacher,* XXXI: 63-67, February, 1938. Shows that an organized program is to be preferred.

19. Wilson, Guy Mitchell. *What Arithmetic Shall We Teach?* Boston: Houghton Mifflin Company, 1926. 149 pp. Reports an investigation of the arithmetic used by persons in various occupations. Suggests a considerably reduced course of study.

CHAPTER 2

THE FUNDAMENTAL OPERATIONS

ADDITION AND SUBTRACTION

The program of the primary grades usually provides for the teaching of the fundamentals of addition and subtraction. If this program has been successfully carried into practice, the teacher in the intermediate grades can proceed at once to more advanced work—more difficult examples—in these operations. However, the primary grades program is far from uniform from school to school because of course of study variations and differences among teachers as to what they undertake to do and how well they do it. Furthermore, pupils differ greatly in ability to learn and in the extent to which they retain what they have been taught. Hence, the first problem of the teacher in any one of the intermediate grades, so far as the fundamentals of addition and subtraction are concerned, is to determine precisely what information, understanding, and skill each pupil possesses. The program to be followed can not be determined entirely in advance; it must be determined in the light of what is revealed by diagnostic tests. The teacher must begin where the pupils are rather than where she or some one else thinks they ought to be.

Finding out where the pupils are. To determine what pupils know and can do in addition and subtraction is much more difficult than many teachers seem to realize. Most teachers who try to find out the degree of competence which pupils have already attained in these topics

administer group tests and conclude that the speed and accuracy with which the pupils solve the examples in these tests give an adequate picture of the progress which the pupils have made to date. That such is not the case is easily shown.

In the first place, the test results do not adequately reveal the pupils' methods of work. Sometimes, a close examination of a pupil's test paper reveals the presence of irregular habits and practices. Thus, the fact that he used a counting process to find the sum of a column may be indicated by the pencil dots sprinkled along the column. But only rarely can a teacher learn fully the habits and methods of work of a pupil by examining his test paper. Many teachers have been greatly surprised at what they discovered in the mental process of their pupils by making a diagnosis of each pupil's habits and methods of solving examples.

Secondly, if a pupil makes a poor showing on a group test, it is only occasionally that the reason for his failure is fully revealed. Some tests are much more diagnostic than are others. For example, an addition test which incorporates the various elements of difficulty in such a manner that the pupil's performance on each may be separately determined is much better than a test which is made up of examples all of which have the same number of addends and the same number of digits in each addend. However, the best of the diagnostic tests may not reveal the fundamental difficulty when pupils fail. The tests may reveal *where* the pupils fail but they may not reveal at all *why* they fail.

Finally, group tests reveal little of a pupil's understanding of numbers and the intricacies of addition

and subtraction. A fourth-grade pupil who has been taught in the primary grades by a method founded upon the drill theory of instruction may react with fair assurance to the combination, $8 + 5$, by giving the sum, 13, but the fact that he does so is no assurance that he *understands* what he may have repeated from memory. He may not even know the nature of addition. Again, he may carry when adding two-digit numbers and he may carry correctly but the carrying process may be a purely mechanical one quite devoid of understanding or appreciation. He may not know why he carries the left-hand figure of the number obtained as the sum of a column rather than the right-hand figure or, for that matter, why he carries at all.

It is essential, then, that the teacher begin the work of the year by studying individually the pupils of her class. The first step may well be the administration of group diagnostic tests. But after these tests have been given and the results have been minutely analyzed, the teacher must study each pupil further and find out what understandings he has, the specific nature of his difficulties, and the general level of the skill which he has attained.

THE TEACHING OF ADDITION

Reviews of the work done in preceding grades.[1] We have seen that it is essential that the teacher first learn to know her pupils as individuals. She must also know intimately the arithmetic program to which each has

[1] For a detailed statement of addition in the primary grades, see the author's *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades*, Chapters 4 and 5.

been subjected in preceding grades. Only by knowing what each pupil has already been taught and by knowing the habits, skills, insights, and understandings which each now possesses, can a satisfactory program for the year be planned.

The addition combinations. We have, first, the 45 fundamental addition combinations. Each of these combinations, except the doubles, yields two addition facts. Thus, the combination which is formed from the numbers 5 and 7 yields the two facts which with the sums may be stated as $\begin{array}{r}5\\ \underline{7}\\ 12\end{array}$ and $\begin{array}{r}7\\ \underline{5}\\ 12\end{array}$. Since there are 9 of these 45 combinations which are doubles, it is obvious that these nine produce but 9 addition facts. Of course, the 36 combinations which are not doubles produce 72 addition facts. This gives in all 81 addition facts. Pupils should be tested for their understanding and mastery of these 81 facts. They may be arranged in miscellaneous order for testing purposes as indicated in Test I.

Review Tests for the Intermediate Grades. Test 1

The 81 Basic Addition Facts

2	1	3	9	1	8	5	7	9
4	5	7	6	2	3	4	7	1
7	2	6	8	1	3	6	2	9
5	8	9	4	7	4	5	6	9
5	8	1	4	9	8	6	7	2
7	6	4	5	3	8	4	6	5

3	4	8	8	5	6	4	5	2
1	7	9	1	9	6	3	8	2
6	2	7	9	6	4	1	7	8
7	1	2	5	8	2	1	9	2
5	4	3	7	6	1	9	3	8
5	9	6	8	3	8	4	3	7
6	3	1	5	8	2	7	1	3
1	8	9	2	5	3	4	6	5
2	4	5	4	2	7	6	5	9
9	8	1	4	7	1	2	6	2
1	4	9	5	3	9	7	4	3
3	6	7	3	2	8	3	1	9

It has been indicated that the teacher should not only determine how readily pupils can react to these addition facts by giving the correct sums but also how well they understand the process of addition which is exemplified by these facts and what their habits of work are. The teacher must spot those who use counting or other circuitous means of finding sums. Counting is important as a means of discovering the sums of the addition facts when these facts are first encountered but if the pupil has formed the habit of counting to find sums, this habit must be broken. It may be broken by re-teaching the facts which are not known, by showing the economy of time and the reduction in errors which accompany the giving of immediate responses, by working with the pupils individually as a means of encouraging

them to give sums quickly, and by practice or drill exercises which place some emphasis upon speed.

Other circuitous responses are frequently found when the methods of work of the individual pupils are studied. For example, a pupil may react to $\begin{array}{r}9\\ \underline{7}\end{array}$ as follows: 10 and 7 are 17; 9 is one less than 10 so 9 and 7 must be one less than 17; then, 9 and 7 are 16. Now, this is a highly intelligent procedure for a pupil who is adding $\begin{array}{r}9\\ \underline{7}\end{array}$ for the first time. To discover that the sum of 9 and 7 is the same as the sum of 10 and 6, or 16, is to make a most important discovery. It shows how such a combination as $\begin{array}{r}9\\ \underline{7}\end{array}$ may be rearranged as $\begin{array}{r}10\\ \underline{6}\end{array}$ for convenience by making use of the idea of 10. But to learn $\begin{array}{r}9\\ \underline{7}\end{array}$ by perceiving its relation to $\begin{array}{r}10\\ \underline{6}\end{array}$ is quite different from making a habit of arriving at the sum of 9 and 7 by this round about method. Eventually, the well-taught pupil will know that 9 and 7 are 16 at once without going through the intermediate steps. Likewise, the pupil who knows that 8 and 8 are 16 may well observe that 8 and 7 is 1 less than 8 and 8 and, hence, must equal 15 as he learns $\begin{array}{r}8\\ \underline{7}\end{array}$ but here, again, he must eventually be able to give 15 at once as the sum of 8 and 7.

Pupils who are exceptionally slow in their responses in addition are usually found to be making use of some kind of roundabout method of arriving at sums. But the teacher dare not assume that slowness means necessarily undesirable methods of work. Some pupils are

slow because they are deliberate in all that they do, because they were born that way. To assume that such pupils are using unapproved methods and to try to force them to greater speed through the use of closely timed tests is ruinous. Since the correctly formed habits are not allowed to function and since there are no other habits to function, such pupils are greatly confused. Therefore, we say again that the teacher must know well the individual pupils and their methods and habits of work. The teacher who knows only the obvious explicit responses of the pupils often misses entirely the key to the situation. The key may lie in the implicit responses and these will not be known unless the teacher in her work with the individual pupils brings them to the surface.

Higher-decade addition. The next important matter for review so far as addition is concerned is higher-decade addition. In higher-decade addition, the pupil adds a one-digit number to a number of two or more digits, usually two digits. For instance, in the example at the right, the pupil, if he adds downward, first makes use of one of the 81 basic facts, $\begin{array}{r}4\\ \underline{8}\end{array}$. The remaining addition steps, however, are instances of higher-decade addition. Obviously the higher-decade combinations which occur in this example, if it is added downward, are $\begin{array}{r}12\\ \underline{5}\end{array}$, $\begin{array}{r}17\\ \underline{7}\end{array}$, $\begin{array}{r}24\\ \underline{2}\end{array}$, and $\begin{array}{r}26\\ \underline{6}\end{array}$.

$$\begin{array}{r}4\\8\\5\\7\\2\\ \underline{6}\end{array}$$

The combinations $\begin{array}{r}12\\ \underline{5}\end{array}$ and $\begin{array}{r}24\\ \underline{2}\end{array}$ have sums in the same decade as is the larger addend. Such higher-decade com-

binations are said to be higher-decade combinations without bridging. But the combinations $\begin{array}{r}17\\ \underline{7}\end{array}$ and $\begin{array}{r}26\\ \underline{6}\end{array}$ have sums in the next higher decade. These are said to be higher-decade combinations *with* bridging. Evidence furnished by Buswell and John[2] indicates that higher-decade addition in which bridging is required is considerably more difficult than is higher-decade addition in which bridging is not required.

Many pupils become proficient in the use of the basic addition facts but have difficulty with column addition because they do not know the higher-decade combinations. Recently a teacher wrote the author to the effect that he had been diagnosing the difficulties of a group of seventh-grade pupils who were having trouble with arithmetic. This diagnosis soon revealed that these pupils could not extend their skill with the fundamental combinations to the higher-decade combinations and that for this reason column addition was almost impossible for them. Many who did not count at all in combining two one-digit numbers readily fell into the counting habit as they added farther down or up a column.

Tests 2 of our review tests includes 90 higher-decade combinations which do not require bridging. These are all of the higher-decade combinations which are found in the 'teens and the twenties, there being 45 in the 'teens and 45 in the twenties.

[2] Buswell, G. T. and John, Lenore. *Diagnostic Studies in Arithmetic.* Chicago: The University of Chicago, 1926, Chapter III.

Review Tests for the Intermediate Grades. Test 2

Higher-Decade Addition without Bridging

14 3	16 2	21 7	24 3	23 5	12 4	13 6	25 3	10 9
27 1	12 6	24 5	15 2	22 1	20 6	20 4	23 1	21 4
26 2	18 1	14 5	12 1	10 1	21 8	25 4	21 1	23 2
20 1	14 2	13 1	10 2	25 2	23 3	21 5	20 3	12 2
10 4	21 6	14 1	21 3	20 2	23 4	26 1	11 7	10 3
22 3	14 4	25 1	23 6	22 2	10 5	11 8	15 3	20 5
12 3	20 7	24 1	10 6	26 3	22 4	15 1	11 1	10 7
24 2	10 8	22 5	11 2	12 5	13 2	12 7	15 4	20 8
16 1	27 2	11 3	17 1	11 5	16 3	13 4	11 4	22 6
20 9	13 3	24 4	28 1	11 6	22 7	13 5	17 2	21 2

Test 3 includes 90 higher-decade combinations in which bridging is required. This test contains the 45 combinations in which one bridges from the 'teens to

the twenties and the 45 combinations in which one bridges from the twenties to the thirties.

REVIEW TESTS FOR THE INTERMEDIATE GRADES. TEST 3
Higher-Decade Addition with Bridging

19	24	17	26	15	28	11	29	12
3	6	7	7	8	3	9	4	8
17	26	19	24	28	12	29	17	19
8	8	4	7	4	9	5	9	5
24	19	26	18	28	16	29	13	24
8	6	9	2	5	4	6	7	9
15	27	15	28	27	29	19	18	25
7	3	9	6	4	7	7	3	5
29	16	28	13	18	25	27	19	28
8	5	7	8	4	6	5	8	8
16	13	29	14	16	28	18	14	16
6	9	9	6	7	9	5	7	8
25	19	21	18	16	25	27	14	17
7	9	9	6	9	8	6	8	3
22	18	25	27	14	29	18	27	18
8	7	9	7	9	1	8	8	9
22	17	26	15	29	23	19	27	17
9	4	4	5	2	7	1	9	5
23	15	26	19	23	28	26	17	29
8	6	5	2	9	2	6	6	3

Practice on higher-decade addition will not be confined to the combinations in the 'teens and the twenties which are included in these two lists. These lists make good review tests because they cover the items most frequently used and constitute a good sample of higher-decade addition combinations since they cover the ground systematically as far as they go. But as the pupil studies higher-decade addition, he should see each of the combinations as intimately related to one of the basic addition facts and he should also see these combinations as families. Thus, the combination, $\begin{array}{r}25\\ \underline{8}\end{array}$, will be seen in its relation to $\begin{array}{r}5\\ \underline{8}\end{array}$ and it will also be seen as a member of a family which includes $\begin{array}{r}15\\ \underline{8}\end{array}$, $\begin{array}{r}25\\ \underline{8}\end{array}$, $\begin{array}{r}35\\ \underline{8}\end{array}$, $\begin{array}{r}45\\ \underline{8}\end{array}$, etc.

Column addition. Higher-decade addition has two major uses. It is used in column addition, as we have seen. This is the more common, and probably the more important use. It is also used in carrying in multiplication. In the example shown, for instance, the pupil must add the 6 which he carries to 28 and then he must add a carried 3 to 42. Thus, this example requires the use of the two higher-decade addition combinations, $\begin{array}{r}28\\ \underline{6}\end{array}$, and $\begin{array}{r}42\\ \underline{3}\end{array}$.

$$\begin{array}{r}649\\ \underline{7}\end{array}$$

Among the tests and practice exercises given to pupils early in an intermediate grade, there should be examples in column addition. These examples should be constructed so as to provide a systematic and desirable distribution of practice on the fundamental addition facts and on the higher-decade combinations. Thus,

Test 4 contains 45 examples in column addition. These 45 examples are constructed so as to provide practice on the 45 addition facts of which the sums are 10 or more, on the higher-decade combinations in the 'teens, and on the 45 higher-decade combinations which require bridging from the 'teens to the twenties. The first example in the set, for instance, gives practice on the basic fact, $\begin{array}{r}9\\ \underline{2}\end{array}$, the higher-decade combination $\begin{array}{r}11\\ \underline{5}\end{array}$, and the higher-decade combination $\begin{array}{r}16\\ \underline{8}\end{array}$. These items do not occur in any other examples of the test. If the purposes of the test are to be realized, the examples must be added downward.

Review Tests for the Intermediate Grades. Test 4

Column Addition

9	2	7	9	6	5	7	8	6
2	8	8	4	6	5	4	4	8
5	5	1	3	1	2	6	5	1
8	9	4	6	9	8	5	7	5

7	8	5	7	6	3	6	5	9
3	2	6	9	7	7	5	7	8
1	7	2	1	6	9	7	7	2
9	4	7	8	7	4	6	1	9

7	4	4	3	6	4	8	3	8
7	6	9	9	9	7	5	8	6
3	8	1	6	3	1	2	8	4
6	7	8	2	4	9	7	6	3

8	6	4	9	5	9	9	9	1
8	4	8	6	8	5	9	3	9
3	4	2	2	4	2	1	4	3
8	7	6	9	3	7	2	9	8

8	9	9	8	8	7	2	5	7
3	7	1	7	9	6	9	9	5
3	2	6	4	1	5	4	5	3
9	9	5	3	8	5	6	5	8

Similar sets may be prepared to provide practice on other higher-decade combinations.[3] We may have a set, for example, to provide practice on all of the 81 basic facts and on the 45 higher-decade combinations in the 'teens. The examples of this set should be used before the examples in Test 4. Then there may be sets, or tests, requiring the use of higher-decade combinations in the twenties without bridging, requiring the use of combinations in which one bridges from the twenties to the thirties, etc. Each of these tests would provide for further practice on the higher-decade combinations and on the basic facts previously tested also. Naturally, as additional sets are constructed, attention will be focused more and more upon the more difficult basic facts, as is done in Test 4, and on the more difficult higher decade-combinations—those which involve bridging.

Higher-decade addition in multiplication. We have seen that the second important use of higher-decade addition is in carrying in multiplication. Of course, all of

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[3] Morton, R. L., *op. cit.*, pp. 141-148.

the higher-decade addition combinations which are used in carrying in multiplication may also be used in column addition. On the other hand, not nearly all of those which are used in column addition are used in carrying in multiplication. The combination, 23 + 4, for instance, is used in the column addition example shown at the right but it is never used in carrying in multiplication for the simple reason that 23 is not the product of any one of the multiplication combinations.

$$\begin{array}{r} 9 \\ 8 \\ 6 \\ \underline{4} \end{array}$$

The higher-decade addition combinations which are used in carrying in multiplication contain two-digit numbers which are products of multiplication combinations and one-digit numbers which are possible carry numbers. It is easy to see what these are.

The two-digit numbers which are the products of multiplication combinations are listed in the first column of Table 1. In the second column of this table are indicated the carry numbers which may be added to the product numbers in the first column. Thus, if the product number is 10, the possible carry numbers are 1, 2, 3, and 4. This means four higher-decade addition combinations, indicated by the number "4" in the third column of the table. They are: 10 + 1, 10 + 2, 10 + 3, and 10 + 4. None of these four combinations requires bridging, a fact which is indicated in the fourth column of the table.

The reader may readily satisfy himself that the largest carry number which may arise from the use of any multiplying figure is one less than the number represented by that figure. Thus, if we are multiplying by 6, the largest possible carry number which we can have is

5; if we are multiplying by 8, the largest possible carry number is 7; etc. Then, to determine what carry numbers may be used with each of the two-digit product numbers given in the first column of Table 1, we examine the product number to see what its largest one-digit factor is and conclude that the carry-numbers run from 1 to one less than that factor. Consider, for example, the product number 24. Its largest one-digit factor is 8. One less than 8 is 7. Then the carry numbers which can be added to 24 in multiplication are 1, 2, 3, 4, 5, 6, and 7. Of course, we also obtain 24 as a product when multiplying by 6, 4, and 3. If the multiplier is 6, the only carry numbers which may be added to 24 are 1, 2, 3, 4, and 5; if the multiplier is 4, the carry numbers are 1, 2, and 3; if the multiplier is 3, the carry numbers are 1 and 2. By selecting 8, the largest one-digit factor of 24, we automatically include all of the carry numbers which may be added to 24 when the multiplier is 6, 4, or 3.

The nature of higher-decade addition. The reader should note that whether higher-decade addition occurs in adding columns or in carrying in multiplication, a one-digit number is added to a two-digit number *as a single act of thought*. When a pupil is learning higher-decade addition, he should think of a higher-decade combination as related to one of the basic addition facts. Thus, when he thinks of $28 + 4$, he sees it as related to $8 + 4$. He knows that $8 + 4 = 12$ and concludes that the sum of 28 and 4 is a number a little larger than 28 which is related to 12. This number is 32. Then 28 and 4 are 32. Eventually, however, he should be able to think of 32 at once when he thinks

TABLE 1. HIGHER-DECADE ADDITION COMBINATIONS USED IN CARRYING IN MULTIPLICATION

Product Number	Carry Numbers	Number of Combinations	Number Requiring Bridging
10	1-4	4	0
12	1-5	5	0
14	1-6	6	1
15	1-4	4	0
16	1-7	7	4
18	1-8	8	7
20	1-4	4	0
21	1-6	6	0
24	1-7	7	2
25	1-4	4	0
27	1-8	8	6
28	1-6	6	5
30	1-5	5	0
32	1-7	7	0
35	1-6	6	2
36	1-8	8	5
40	1-7	7	0
42	1-6	6	0
45	1-8	8	4
48	1-7	7	6
49	1-6	6	6
54	1-8	8	3
56	1-7	7	4
63	1-8	8	2
64	1-7	7	2
72	1-8	8	1
81	1-8	8	0
	Total	175	60

of $28 + 4$ without other time-consuming thoughts.

When a higher-decade combination is encountered in column addition, the one-digit number is seen but the two-digit number is not seen. Thus, in adding downward the example shown, 6 must be added to 14. The 6 is visible but the 14 is not visible. When a higher-decade combination occurs in carrying in multiplication, neither the one-digit number nor the two-digit number is visible. Both must be held in mind until the addition is accomplished. And, we repeat, this addition is accomplished as a single act of thought. There is no carrying. One does not think, in the above example, "4 and 6 are 10, write 0 and carry 1, 1 and 1 are 2." One thinks, simply, "14 and 6 are 20."

$$\begin{array}{r}5\\9\\\underline{6}\end{array}$$

Many pupils fail, for lack of proper training, to bridge the gap between the fundamental addition combinations and higher-decade addition. Many resort to counting when they have proceeded but a little distance in adding a column. In multiplication, they often write down the carry number and then combine it with the next product number by counting or by writing both the product number and the carry number at one side and adding as in an ordinary addition example. This practice is time-consuming and error-producing. Higher-decade addition must be mastered and mastered thoroughly if either column addition or multiplication is to be done efficiently.

Carrying. Carrying should be taught in such a manner that the pupils understand *why* they do what they do. Many teachers in the primary grades and in the intermediate grades, simply *tell* the pupils what to do

when carrying is necessary. Here, as in many other places in arithmetic, *telling is not teaching*. If pupils are to be intelligent about arithmetic, if they are to attain the ends described in Chapter 1, they must learn arithmetic as something more than a bag of tricks and a collection of devices.

It has been suggested[4] that the first lessons in carrying may well be illustrated by the use of coins. Since our number system and our money system are both decimal systems, cents and dimes lend themselves admirably to an explanation of carrying when two-digit numbers are added. However, the pupils should soon begin thinking of carrying in terms of numbers rather than in terms of coins.

Whether the teacher in an intermediate grade finds it necessary to teach carrying because it has not been taught, or to re-teach it because it has been taught poorly, or merely to review it, numbers should be broken down and the meaning of the carrying operation made clear in some such manner as the following:

47. This is 40 plus 7 or 4 tens plus 7 units.
26. This is 20 plus 6 or 2 tens plus 6 units.

First, the addition in the broken down form should be performed and the example made to appear as follows:

47. This is 40 plus 7 or 4 tens plus 7 units.
26. This is 20 plus 6 or 2 tens plus 6 units.
60 plus 13 or 6 tens plus 13 units.

Then, the pupils readily recognize that 13 units is the same as 1 ten and 3 units and the work is rewritten or modified to read as follows:

[4] Morton, R. L., *op. cit.*, pp. 152-156.

47. This is 40 plus 7 or 4 tens plus 7 units.
26. This is 20 plus 6 or 2 tens plus 6 units.
73. This is 70 plus 3 or 7 tens plus 3 units.

After pupils have learned how to carry, sets of examples which give practice in carrying should be prepared. These examples should be constructed so as to require that various numbers be carried. Too often in the pupil's early experience with carrying, "carry" means "carry 1." Also, these examples should provide for a desirable review of the fundamental addition facts and of the higher-decade combinations. Review Test 5 illustrates briefly the preparation of such examples. In this test, the pupil is required to carry 1 five times, 2 four times, and 3 three times. The digits are selected so that no fundamental fact or higher-decade combination occurs twice in the 12 examples of the test.

Review Tests for the Intermediate Grades. Test 5

Practice in Carrying

46	12	78	29	41	37
27	68	64	48	30	19
35	53	46	29	66	48
23	34	75	57	65	38
34	16	57	69	38	97
84	74	68	39	73	90
52	45	30	87	49	23
36	47	92	75	74	58

Getting better acquainted with the number system. In reviewing carrying in the intermediate grades the teaching should give the pupils an opportunity to get

better acquainted with the number system. Learning the number system is a slow process. It can be well begun in the primary grades but it can not be completed there. As the pupil passes through the grades of the elementary school, he should gradually and constantly learn to know numbers and the number system better. Some increase in the pupil's understanding of numbers will come incidentally as the various topics of arithmetic are studied but there can be little assurance that satisfactory progress will be made unless the teacher makes this matter the object of very definite attention.

Many pupils, when they come to the study of decimal fractions and decimal mixed numbers, do not readily grasp the significance of figures at the right of the decimal point simply because they do not understand the significance of figures at the left of the decimal point. In other words, they can not understand decimal fractions because they do not understand whole numbers.

Our whole numbers, of course, are decimal numbers; that is, they are expressed in terms of 10 and powers of 10. Thus, the number, 537, is a decimal number although not a decimal fraction or a decimal mixed number. The pupil in the intermediate grades should be able readily to take numbers apart, as it were, to see such a number as 537 as 5 hundreds and 3 tens and 7 units, or as $500 + 30 + 7$. Later, he should see it as $5 \times 10^2 + 3 \times 10 + 7$ and such a number as 6293 as $6 \times 10^3 + 2 \times 10^2 + 9 \times 10 + 3$. Finally, when the subject of exponents has been studied in algebra, the pupil may see 6293 as $6 \times 10^3 + 2 \times 10^2 + 9 \times 10^1 + 3 \times 10^0$.

The writer sometimes finds it difficult to get teachers and prospective teachers to see that the number system may be hard for pupils to understand. If the reader will work for awhile with a system which is not a decimal system, perhaps he will appreciate better the difficulties which pupils experience. Suppose, for example, that we had a duodecimal system, a system whose radix or base is 12 instead of 10. Then 12 in our decimal system would become 10 in the duodecimal system, our 13 would become 11, our 14 would become 12, our 20 would become 18, our 24 would become 20, our 37 would become 31, etc. Our numbers, in other words, would be expressed in terms of 12 and powers of 12. Such a number as 537 in the duodecimal system would mean $5 \times 12^2 + 3 \times 12 + 7$. Thus, 537 in the duodecimal system would be equivalent to 763 in our decimal system. To convert a decimal number into a duodecimal number number, it must be expressed in terms of 12 and powers of 12. Thus, the decimal number, 100, would be 8 twelves and 4, or 84. The decimal number, 452, which is 37 twelves plus 8, would be $3 \times 12^2 + 1 \times 12 + 8$, or 318 in the duodecimal system. If this is difficult for the teacher, let him remember that the decimal system is probably more difficult for the pupil, for he is intellectually less mature.

Zero difficulties. It is not uncommon to find pupils who are uncertain as to what they should do with zeros in addition. One reason for this uncertainty is that the meaning and the uses of zeros are not taught by teachers after the time when an effort is made to teach their uses in the primary grades. Then, too, zero combinations are often taught and drilled upon separately in the pri-

mary grades when they never occur in the world of practical affairs in this separate, isolated form. Thus, a pupil who is taking a test on the fundamental addition facts may find in the test such items as $\begin{array}{r}6\\ \underline{0}\end{array}$ or $\begin{array}{r}0\\ \underline{6}\end{array}$. Obviously, these occur in examples like $\begin{array}{r}46\\ \underline{30}\end{array}$ or $\begin{array}{r}208\\ \underline{162}\end{array}$ but not in the separate form in which they may appear in tests. It hardly seems likely that the pupil will be well prepared for real uses of zeros in addition if his training and practice have been confined to the unreal form indicated.

The history of 0 shows that this symbol came into use much later than did the other Hindu-Arabic numerals. Dickey points out that at least seven centuries elapsed after the introduction of the Hindu-Arabic numerals into Europe before 0 appeared.[5] It seems that 0 first appeared in the ninth century, A.D. It is noteworthy, also, that when the symbol for zero appeared, it was used as a means of indicating a void, that is, the lack of any of an order rather than as a symbol for nothing. Thus, 0 became the "sign for the empty column," as Wheat says.[6] Wheat took up the discussion of zero which Dickey had started and pointed out that the function of zero, so far as elementary school arithmetic is concerned, is that of a place holder. If, for example, we desire to have the numeral 3 represent 3

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[5] Dickey, John W. "Much Ado about Zero." *Elementary School Journal,* XXXII: 214-222, November, 1931.

[6] Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937, pp. 71-80.

tens rather than 3 units we use a zero to keep the 3 i
its place and write 30.[7]

The conception of zero as a place holder means th
complete abolition of the zero combinations in separate
isolated form. Surely, there can be no justification fo:
drill on zero combinations in this form. However, one
can hardly say that there are no zero combinations. In
such an example as $\begin{array}{r}547\\ \underline{302}\end{array}$, the pupil encounters a zero
combination and must know what to do about it. What
he should learn to do eventually is to neglect the zero
and add as if it were not there, to treat it as a place
holder, in other words. But if, after thinking "7 and 2
are 9," the pupil thinks in like manner "4 and 0 are 4,"
or "4 and there is nothing more, so it is 4," no harm
is done. Whether the pupil thinks, "the 0 merely holds
the 3 in its place so all I have is 4," or "4 and 0 are 4," he
must understand that in neither case is there anything
to add to 4 so all we have to do is to write "4" in the
sum.

Grosswickle adds one more article to the series about
zero.[8] He suggests that in multiplication the pupil has
no need for zero combinations in which the zero is in
the multiplier position but that he does have need for
zero combinations in which the zero is in the multipli-
cand position. This point will receive further considera-
tion in the next chapter. This again suggests that for
pupils in the grades the appearance of zero combina-
tions in examples is a very real sort of thing and can

[7] Wheat, Harry G. "More Ado about Zero." *Elementary School Journal*, XXXII: 623-627, April, 1932.

[8] Grosswickle, Foster E. "Still More Ado about Zero." *Elementary School Journal*, XXXIII: 358-364, January, 1933.

not be satisfactorily disposed of at this stage of the pupil's progress by the explanation that the sole function of zero is that of a place holder.

Wheat admits that zero becomes a numeral like the other numerals in later mathematics. It is used to indicate a position on a scale, as a thermometer scale, and as a division point between positive and negative numbers.[9] Grosswickle adds that zero sometimes takes on the same function as any other natural number in decimal fractions.[10] Thus, .500 does not mean the same thing as .5 or .50. The decimal fraction, .500, means that a measurement or a calculation has been carried to three places and that the result is correct *to the nearest thousandth* whereas .50 means "correct to the nearest hundredth" and .5, "correct to the nearest tenth," only.

There is excellent reason to believe that in the hands of a skillful teacher, the zero combinations become the easiest of all the combinations instead of the most difficult. It is quite right that they should be the easiest. There is nothing intrinsically hard about them when the meaning of zero is understood.

In solving addition examples having addends of unequal length, some pupils are disturbed over the necessity of jumping the blank spaces. Of course, they need to learn to ignore the empty spaces and to add in each column the numbers which the figures of that column represent. The situation is similar to that which the pupil finds when the columns contain zeros. Some teachers

$$\begin{array}{r} 42 \\ 5671 \\ 384 \\ 9 \\ 27 \\ \hline \end{array}$$

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[9] Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937, p. 80.

[10] Grosswickle, F. E., *loc. cit.*

go so far as to fill in the empty spaces with zeros but this is quite unnecessary if the figures are of uniform size and spacing and are written in straight columns.

New work in addition. Whether all of the work in addition so far discussed will be undertaken in the primary grades, that is, the first three grades of the elementary school, is a question. In some schools, all that has been considered is done in the primary grades but in other schools, the program is considerably less extensive. There is a current tendency to do less in the primary grades than was done in preceding years. Whether this tendency will turn out to be a real reform, is not yet known.

Work in addition beyond that so far discussed is not really new; it is largely an extension and an elaboration of that already done. However, it is desirable that each of the elements of skill developed in the primary grades be carried to a higher level in the intermediate grades. In any phase of addition, the fourth-grade pupil should be more proficient than the third-grade pupil, the fifth-grade pupil more proficient than the fourth-grade pupil, etc. But new work in addition will be largely an extension of the addition abilities already discussed. The columns will be longer, there will be a greater number of digits in an addend, and there may be examples containing addends of more greatly varying length but there will be no essentially new abilities required of the pupil.

Further practice in higher-decade addition. Other things being equal, the longer the columns in an addition example, the greater the variety in the higher-decade addition combinations which may be encountered.

But it should be no more difficult for a pupil to react to one higher-decade addition combination than to another which is similar to it but in a lower decade. The combination, $77 + 5$, should be no harder than $27 + 5$ or $37 + 5$ although it is much less frequently encountered. It has already been said that these higher-decade combinations are related to the basic addition facts (in this case, to $7 + 5$), and that these higher-decade combinations occur in families and are, therefore, intimately related to each other. The pupil should perceive this relationship and profit by it.

We have seen that there are 45 higher-decade addition combinations in each decade which have sums in the same decade as the larger addend and that there are 45 in each decade whose sums are in the next higher decade. This means 17 groups of 45 higher-decade combinations each, or 765 combinations, whose sums are less than 100. It is not necessary that all these 765 combinations be woven into sets of practice examples. Because of the close relationships involved, we can expect a large measure of transfer. But in constructing examples for practice, we should provide carefully for a wide variety of practice. This is done in the examples of Set 1.

The five examples of this brief set give practice on certain combinations the sums of which are in the forties and fifties. Naturally, a review is provided for several of the fundamental facts and for a number of higher-decade combinations the sums of which are in the 'teens, twenties, and thirties. In providing this review, no fundamental fact or higher-decade combination of a given type is included more than once. For

PRACTICE EXAMPLES IN ADDITION. SET 1

Variety in Higher-Decade Addition

878	754	669	448	586
192	229	686	837	469
244	187	357	774	599
167	396	596	428	722
338	478	298	976	319
486	559	537	468	117
359	617	365	274	146
226	241	551	356	355

instance, since the right-hand column of the first example provides for the addition of 10 and 4 (adding downward) neither 10 + 4, nor 20 + 4, nor 30 + 4 occurs again in that example or in any other example of the set. Likewise, no combination of a given type the sum of which is in the forties or fifties occurs more than once in these five examples. Many drill exercises neglect this matter of distribution of practice. One author, in a series of five sets of examples in column addition, includes the combination, 23 + 3, five times but does not include the combination 23 + 9, at all. The same series of examples provides for the addition of 2 + 21 four times, 8 but once, and 1 and 3 not at all.[11]

Attention span. Many pupils who know their pri-

[11] Morton, R. L. "Higher-Decade Addition in Some Recent Drill Devices." *Journal of Educational Research*, XV: 104-110, February, 1927.

mary facts well and who are skillful in higher-decade addition will have difficulty in adding long columns such as that shown at the right. The difficulty is explained in terms of the psychology of *attention span*. Adding downward, the well-trained pupil will think, "12, 17, 24, 26, 32, 35, 44" but may be unable to keep his attention on the example until he has done the eleven additions required. His attention may falter before he reaches the 44, he may go a step or two farther, or he may do all eleven additions without difficulty or hesitation, but if the example is made long enough the limit of his attention span is sure to be reached.

$$\begin{array}{r} 4 \\ 8 \\ 5 \\ 7 \\ 2 \\ 6 \\ 3 \\ 9 \\ 2 \\ 4 \\ 1 \\ \underline{6} \\ 57 \end{array}$$

Pupils differ greatly in attention span, as in other traits. Some seem to be able to do only three or four additions without a pause; others may do twelve, fifteen, or even twenty. The author has found that under ordinary conditions, his attention span will easily carry him over twenty-five or thirty additions. Courtis states that "Experiment shows that most children can add steadily for six additions."[12] The teacher should study the individual members of a class in an effort to determine the attention span of each of them.

The teacher can usually recognize attention span troubles without difficulty. "If the child hesitates at regular intervals in a column, if he goes over and over a given addition and is apparently unable to think at all, if he gives up in the middle of a column and begins

[12] Courtis, S. A. *Courtis Standard Practice Tests in Arithmetic, Teachers Manual.* Yonkers, New York: World Book Company, 1917, p. 38.

again, the difficulty is almost sure to be one of this nature."[13]

Apparently, one's attention span can be improved through practice. The author, in working with students in teacher-training courses, has found that the number of additions performed in a single span of attention increased appreciably in a brief period of practice. In the pupil's early work in addition, the teacher should not permit columns to be longer than those which the pupil can add within a single span of attention. Then the columns should be permitted to increase in length gradually as the pupil's attention span is increased.

The pupil should be taught what to do when he reaches the limit of his attention span before he reaches the end of a column. It seems best for him to repeat to himself the partial sum obtained while he holds his place in the column with his pencil, takes his eyes from the paper and makes some slight bodily movement. "Teach the child to recognize the difficulty when it occurs, to avert his attention momentarily by lifting his eyes and taking a deep breath, to keep his place in the column by pointing with his pencil, and to remember correctly the partial sum. Children are apt to get entangled in the situation and to go over helplessly the same addition so long that when the crisis occurs and the mind clears, the additions which have been made previously are totally forgotten."[14]

It is not unusual to find pupils who are failing to make satisfactory progress in addition in the interme-

[13] Courtis, S. A., *op. cit.*, p. 39.
[14] Courtis, S. A., *op. cit.*, p. 39.

diate grades because of attention span difficulties. Worse, still, is the fact that the teacher is often unaware of the nature of the pupils' difficulties and, therefore, is unable to help them. In planning lessons in addition, we must take very definite account of this phase of addition ability.

Speed versus accuracy in addition. It is not unusual to find teachers who stress speed in addition as if speed were the primary objective. This emphasis upon speed to some extent results from the use of standardized attainment tests and practice exercises. Speed is an important *symptom* of mastery but it is not nearly so important as is accuracy. Accuracy is the primary objective. But the time which a pupil requires to solve an addition example may indicate to the teacher that his addition habits are unsound. If he is slow, he may be employing elaborate and roundabout methods instead of the economical procedure of straightforward addition.

The teacher should remember that the same habits which yield accurate results will usually produce speed, and *vice versa.* If the pupil is well taught, and if he understands what he is doing, his speed will be largely determined by his reaction time. Therefore, lack of speed is not in itself necessarily a fault, but it may be indicative of serious faults, such as the use of indirect methods which reduces speed and causes errors. The habits of a slow pupil should be investigated thoroughly; if they are correct, and if his results are accurate, his teacher need not be much concerned about his speed.

Checking results. The practice of checking results in

all computation is of fundamental importance. This practice should be started in the primary grades. Teachers should insist that pupils follow the practice in higher grades. Unfortunately, pupils often are supplied with textbooks containing answers and, as a consequence, they have little incentive to check results. Answers may be justified for some portions of arithmetic, particularly that taught in the upper grades, but in straightforward computation with integers they have no place. It is doubtful whether they are justified at all in the intermediate grades. It is important, then, that the pupils be trained to check all of their computations as the one and only means of assuring themselves that their answers are correct.

The checks which are usually employed are: (1) repeating the addition in the same direction; (2) repeating the addition in the opposite direction. Repeating the addition in the same direction is the more easily applied check, for it is simply a repetition of what the pupil has already done. The objection to this check lies in the fact that if the pupil has made an error, he tends to repeat that error on the second and later addings. Since the sum, up to the point where the error was made, is the same each time, adding the number represented by the next digit is likely to give the result previously secured, on account of the association which has been established, incorrectly of course, between that result and the previous sum.

It is better, then, to check by adding in the opposite direction. If the first adding is downward, check by adding upward. If the first adding is upward, check by adding downward. Adding in the opposite direction

gives partial sums different from those secured by the first adding and thus makes it improbable that a former error will be repeated.

The check "casting out nines," is sometimes recommended. This check is of doubtful value in addition. It should not be used in the intermediate grades for any of the fundamental operations.

Addition in problems. Our discussion of addition has included: (1) the fundamental combinations; (2) higher-decade combinations; (3) carrying; (4) the zero difficulties; (5) addends of unequal length; (6) attention span; and (7) checking. We have discussed these details with little reference to the uses of addition in problem solving. This plan has been followed merely for convenience. This does not mean that the pupils must be subjected to continuous practice on an operation for which they can see no need. Indeed, each new phase of the work in addition should be introduced through the medium of a life-like problem in order that the pupils may see a need for acquiring skill in that particular phase. Furthermore, pupils should have frequent opportunities to apply their skills in the solution of problems in which they are interested, particularly problems growing out of activities in which the pupils are engaged. A more detailed discussion of problems and problem solving will be provided in a later chapter.

THE TEACHING OF SUBTRACTION

The relative complexity of addition and subtraction. Neither ability in addition nor ability in subtraction is a single, simple ability but is a complex of several abili-

ties. In addition, one must know the primary facts, must be able to extend these facts to higher decades without and with bridging, must know what to do with zeros, must be able to carry, must be skillful in adding addends of unequal length, must know what to do when he reaches the limit of his attention span, and must know how to check his answers. Subtraction ability, however, is a composite of fewer skills. One must know the primary facts, must know what to do with zeros, must be able to "borrow," must be able to solve examples in which the subtrahend has fewer digits than the minuend, and must know how to check his answers. Attention span may be a problem in subtraction if the examples contain very large numbers but this problem is less serious in subtraction than in addition. Examples made up of numbers of unequal length are also less difficult in subtraction than in addition. Higher-decade addition has its counterpart in higher-decade subtraction but whereas higher-decade addition is absolutely essential in column addition, higher-decade subtraction is not needed at all in an ordinary subtraction situation but is used only in short division, as we shall see.

The problems pertaining to the teaching of subtraction are, then, fewer and simpler than those pertaining to the teaching of addition. In a subtraction example, there are but two numbers to deal with—the minuend and the subtrahend.

Methods of subtraction. When subtraction is begun, the teacher is obliged to choose one of the various subtraction methods. There are five of these, the first four of which are rather well known. They may be listed as follows:

1. The take-away-borrow method
2. The take-away-carry method
3. The addition-borrow-method
4. The addition-carry method
5. The complementary method

The same example will be used to illustrate each of these five methods.

If subtraction is done by the take-away-borrow method, the example is solved as follows: "8 from 15, 7; 7 from 13, 6; 4 from 9, 5; 9 from 15, 6; 2 from 3, 1." In the first step, the 5 units are increased to 15 units and to compensate for this, the 4 tens are reduced to 3 tens. For convenience, one may think of one of the 4 tens being changed to units, leaving 3 tens and making 15 units. This method is often called *the method of decomposition* since the 4 tens are broken up or decomposed and one of the tens is changed to units.

$$\begin{array}{r} 46045 \\ \underline{29478} \\ 16567 \end{array}$$

If the take-away-carry method is used, one thinks: "8 from 15, 7; 8 from 14, 6; 5 from 10, 5; 10 from 16, 6; 3 from 4, 1." Here again the 5 units are increased to 15 units but to compensate for adding 10 to the minuend in this manner one also adds 10 to the subtrahend and adds it in at the tens' place, making the 7 tens 8 tens. Because the same number is added to both the minuend and the subtrahend this method is often called *the method of equal additions*. In practice, the pupil who uses this method seldom thinks of adding the same number to both the minuend and the subtrahend but thinks simply, "8 from 15, 7, carry 1 (since the 5 became 15); 8 from 14, 6, carry 1;" etc. Note that where there is no

$$\begin{array}{r} 46045 \\ \underline{29478} \\ 16567 \end{array}$$

borrowing (or carrying), the first and the second methods are identical. Each becomes simply a take-away method.

In the third method, one borrows as in the first method. But here he finds the number which, added to the subtrahend, produces the minuend instead of the number which remains if the subtrahend is taken from the minuend. In solving this example by the addition-borrow method, one thinks: "8 and 7 are 15; 7 and 6 are 13; 4 and 5 are 9; 9 and 6 are 15; 2 and 1 are 3." Note that the figure which goes in the answer is the second one thought or spoken in each step.

$$\begin{array}{r} 46045 \\ \underline{29478} \\ 16567 \end{array}$$

The fourth method is like the second method in that one carries and like the third method in that one adds. The example is solved as follows: "8 and 7 are 15; 8 and 6 are 14; 5 and 5 are 10; 10 and 6 are 16; 3 and 1 are 4." Again, the second figure in each step is the answer figure. Note that this method is identical with the third method if there is no borrowing (or carrying). Both the third and the fourth methods are sometimes referred to as *the Austrian method* although this term is more frequently applied to the fourth method than to the third.

$$\begin{array}{r} 46045 \\ \underline{29478} \\ 16567 \end{array}$$

The complementary method may be used with either a borrow process or a carry process. If the borrow process is used, the example would be solved as follows: "8 from 10, 2, and 5 are 7; 7 from 10, 3, and 3 are 6; 4 from 9, 5; 9 from 10, 1, and 5 are 6; 2 from 3, 1." If the carry process is used one thinks as he solves the example: "8 from 10, 2, and 5 are 7; 8 from 10, 2, and 4 are 6; 5 from 10, 5; 10

$$\begin{array}{r} 46045 \\ \underline{29478} \\ 16567 \end{array}$$

from 10, 0, and 6 are 6; 3 from 4, 1." This method does not require the learning of subtraction facts beyond those having 10 as a minuend. One first finds the complement, with respect to 10, of the number represented by the subtrahend figure and then adds to this complement the number represented by the minuend figure. When popular, this method was usually used with a carry process rather than with a borrow process although both were used. Obviously, this method is more involved than is either of the others since it requires two steps for many answer figures instead of one.

The complementary method is seldom mentioned in discussions of subtraction. Concerning the other four methods, however, there are two principal disputes. The first dispute has to do with the relative merits of borrowing and carrying, that is, with the relative merits of decreasing the number represented by the next minuend digit and increasing the number represented by the next subtrahend digit. This dispute evidently applies to methods 1 and 3 as opposed to methods 2 and 4. The second dispute has to do with the relative merits of teaching subtraction as a take-away process and as an addition process. In this dispute, methods 1 and 2 are allied against methods 3 and 4.

The first of these questions has been argued for many years and we are hardly any nearer a definite answer to the question today than we were many years ago. A few limited investigations have resulted in a trace of evidence on each side of the question.[16] Logi-

[16] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company, 1937, pp. 181-185.

cally, one point of view can be supported neither more nor less successfully than the other. The best thing for the teacher in the intermediate grades to do is to accept her pupils as they come to her and to make no effort to change the subtraction methods which they have already acquired. *The teacher must be skillful in the use of all the subtraction methods and able to adapt herself to pupils who may have learned a method which she herself did not learn in the grades.* It is better for the teacher to change her habits than to ask the pupils to change theirs unless there is clearly evidence favoring the change.

The author has adopted the borrowing method for beginners rather than the carrying method for two reasons; (1) it is the method in most common use in this country at the present time; and (2) it seems to be easier to rationalize for the pupils the borrowing process than the carrying process.

The more recent of the two disputes is that as to whether subtraction should be taught as subtraction or as addition. Two reasons have been given for teaching subtraction as an additive process. First, the addition method makes use of the addition combinations, meaning that the pupil need not learn subtraction combinations as such. Secondly, the addition method supposedly resembles making change. As to the first reason, it may be pointed out that if the addition combinations and the subtraction combinations are taught together as teaching units, it is hardly any more difficult for the pupil than if the addition combinations alone are taught and then these combinations are restated in a

different form for subtraction.[16] Furthermore, if both addition and subtraction are taught as such, the pupils seem to get a broader view and a better understanding of the combinations.

As to the second argument, in making change one does not ordinarily subtract at all. Neither does he add. That is, he neither takes away the amount of the purchase from the amount tendered in payment nor does he add to the amount of the purchase that which will make the amount tendered in payment, as in additive subtraction. Rather, he *counts* in terms of our monetary units. Thus, if a clerk is handed a five-dollar bill in payment for a purchase amounting to $2.68, he will ordinarily think as he picks the appropriate coins and bills from the till, or as he counts the change out to the customer, "268, 70, 75, 3, 4, 5." Practices followed in making change do not constitute an argument for either method of teaching subtraction.

The author has suggested the use of a take-away method rather than an additive method for beginners for several reasons which may be stated briefly as follows:

First, the take-away method represents what is ordinarily thought of when reference is made to subtraction. This is indicated by the etymology of the word. "Subtraction" comes from the Latin, *subtractus,* a participial form of *subtraho* meaning literally, "draw from under." The same meaning stands today in ordinary affairs. Depreciation in the value of a house, or a car, is not thought of as an amount to be added to the present sale

[16] Morton, R. L., *op. cit.,* chapter 4.

value. Contemplated expenditures are deducted from what we have, not added to what we shall have left. Reductions in marked-down sales represent what the customer saves out of the original price rather than what should be added to the present price to produce the list price. To be sure, some situations demand the "how-much-more" idea as in the case of the man who has $850 and wants to know how much more he needs to buy a car costing $1095, but even here, he is likely to determine the amount needed by "subtracting" $850 from $1095, rather than by thinking, "what amount added to $850 will make $1095?" The addition method is not subtraction but is a substitute for subtraction.

Secondly, if the take-away method is employed, an example can be checked by adding the remainder to the subtrahend and comparing the sum with the minuend. This valuable check is lost, or nearly so, if the addition method is used. In additive subtraction, the check consists of adding again the subtrahend and the remainder in the same order, or, better, in the reverse order.

Finally, as already stated, the take-away method is the method in most general use. The take-away-borrow method seems to be more generally used than all other methods combined.

Again, we would urge that the teacher be thoroughly conversant with the two addition methods as well as with the two take-away methods whether she has occasion to teach them or not. An intermediate grade class may well include pupils who have learned subtraction methods other than that taught in the school in which they now find themselves. In order that the teacher may help these pupils to locate their errors and

otherwise guide them in their learning, she must be able to adapt herself readily to any subtraction method which may be found in use.

The subtraction combinations. We have seen that there are 45 fundamental combinations in addition if the zeros are not included and that these 45 combinations yield 81 addition facts. The number of combinations and facts in subtraction is the same as the number in addition. It has been suggested that the addition facts and the subtraction facts be taught together as *teaching units.*[17] Thus, the combination, 4 and 6, yields two addition facts and two subtraction facts, as follows:

$$\begin{array}{r} 4 \\ \underline{6} \\ 10 \end{array} \qquad \begin{array}{r} 6 \\ \underline{4} \\ 10 \end{array} \qquad \begin{array}{r} 10 \\ \underline{4} \\ 6 \end{array} \qquad \begin{array}{r} 10 \\ \underline{6} \\ 4 \end{array}$$

These four would be taught together as a teaching unit.

Review Test 6 includes the 81 subtraction facts without zeros just as Review Test 1 included the 81 addition facts.

Review Tests for the Intermediate Grades. Test 6

The 81 Basic Subtraction Facts

$$\begin{array}{r} 6 \\ \underline{5} \end{array} \quad \begin{array}{r} 10 \\ \underline{7} \end{array} \quad \begin{array}{r} 15 \\ \underline{6} \end{array} \quad \begin{array}{r} 3 \\ \underline{2} \end{array} \quad \begin{array}{r} 11 \\ \underline{3} \end{array} \quad \begin{array}{r} 9 \\ \underline{4} \end{array} \quad \begin{array}{r} 14 \\ \underline{7} \end{array} \quad \begin{array}{r} 10 \\ \underline{1} \end{array} \quad \begin{array}{r} 12 \\ \underline{5} \end{array}$$

$$\begin{array}{r} 10 \\ \underline{8} \end{array} \quad \begin{array}{r} 15 \\ \underline{9} \end{array} \quad \begin{array}{r} 9 \\ \underline{6} \end{array} \quad \begin{array}{r} 13 \\ \underline{9} \end{array} \quad \begin{array}{r} 10 \\ \underline{5} \end{array} \quad \begin{array}{r} 10 \\ \underline{2} \end{array} \quad \begin{array}{r} 16 \\ \underline{9} \end{array} \quad \begin{array}{r} 2 \\ \underline{1} \end{array} \quad \begin{array}{r} 6 \\ \underline{2} \end{array}$$

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[17] For a detailed statement of subtraction in the primary grades, see Morton, R. L., *op. cit.,* Chapters 4 and 6.

14	14	9	13	4	13	7	12	14
8	5	2	7	2	8	3	6	9
9	17	11	4	13	10	12	8	7
1	9	7	1	6	4	4	7	4
11	8	18	12	14	5	9	12	15
5	6	9	7	6	4	5	3	8
11	4	9	7	10	9	8	16	6
4	3	3	6	6	8	5	7	4
3	7	16	13	11	8	6	12	5
1	5	8	4	9	3	3	8	2
15	7	11	10	7	13	5	6	8
7	1	8	9	2	5	3	1	4
9	8	11	8	11	17	10	5	12
7	1	6	2	2	8	3	1	9

There should also be a review test in which the subtraction facts are incorporated in examples without borrowing. Such a test can well include the entire 100 subtraction facts—the 81 which appear in Test 6 and the 19 zero facts. In Review Test 7, each of the 100 subtraction facts will be found and 35 of the 55 whose minuends do not exceed 9 occur twice each. These 35 facts are the most difficult of the 55 facts the minuends of which do not exceed 9. No borrowing is required in this test.

Review Tests for the Intermediate Grades. Test 7

Subtraction without Borrowing

1457	1390	1218	1189	1389	1097	1078	1097	1258
806	850	517	904	562	465	638	240	927
1236	1769	1648	1849	1775	1426	1196	1274	1368
413	917	935	947	870	904	603	620	402
1124	1184	1659	1496	1083	1489	1295	1623	1197
521	851	720	780	812	532	731	810	756
1493	1097	1257	1513	1383	1179	1326	1074	1585
611	192	834	703	940	306	704	512	624
1089	1068	1162	1087	1256	1548	1516	1127	1376
763	320	212	903	352	903	806	415	645

Subtraction with borrowing. A plan for rationalizing the operation of borrowing in subtraction should be similar to the plan used for rationalizing the operation of carrying in addition. Coins may well be used in the first lessons, but here again the pupils should soon reach the place where they see the operation of borrowing in terms of the number system rather than in terms of a money system. The meaning of the borrowing operation can be made clear in the following manner.

45. This is 40 plus 5 or 4 tens plus 5 units.
28. This is 20 plus 8 or 2 tens plus 8 units.

Since 8 units cannot be subtracted from 5 units, we can change one of the 4 tens to units, making 15 units and

leaving 3 tens. Then the example may be rewritten as follows:

45. This is 30 plus 15 or 3 tens and 15 units.
28. This is 20 plus 8 or 2 tens plus 8 units.
17. This is 10 plus 7 or 1 ten plus 7 units.

The last item written, of course, is the 17 at the left. This is written after the pupil has observed that 10 plus 7, or 1 ten plus 7 units, is the same as 17.[18]

Review Test 8 requires borrowing and makes provision for practice on each of the 100 subtraction facts. This is done in 25 examples each of which includes a four-digit minuend.

REVIEW TESTS FOR THE INTERMEDIATE GRADES. TEST 8

Subtraction with Borrowing

8743	7380	2657	6000	8461	6347	6200	5123	6402
7408	6717	743	2179	8298	5068	523	1064	3985
3008	7155	9530	4002	9870	8522	4752	7852	7094
1462	3457	3975	1966	5798	4039	3523	988	2397
5930	8720	7761	3581	6428	5041	8754		
5041	6892	2484	2611	1376	4052	5016		

New work in subtraction. New work in subtraction in the intermediate grades will consist largely of increasing the pupil's skill in processes with which he has already become acquainted. His approach to subtraction examples should be characterized by greater and

[18] Morton, R. L., *op. cit.*, pp. 190-192.

greater assurance, his speed should increase, and errors should become less numerous.

Naturally, the examples will become longer, thus placing a greater demand upon attention span. However, there need not be a great deal of practice on examples having large numbers as minuends and subtrahends for large numbers do not occur frequently in the affairs of pupils in the intermediate grades. The examples of Set 1 contain five-digit numbers. These examples, again, are constructed so as to include each of the 100 subtraction facts.

Practice Examples in Subtraction. Set 1

Subtraction of Larger Numbers

39074	54176	22010	97243	31104
26581	47329	14307	16008	15426
56958	36134	92345	83105	21981
55389	32199	77456	46623	6418
93682	58181	37275	84954	61935
46003	43899	13867	80257	12579
80247	98360	93196	67059	53969
76522	95409	72508	41388	39170

Likewise, sets of examples may be constructed with minuends of six or seven digits. Of course, it is not necessary to have the same number of digits in each minuend in a set of practice examples; it is better, perhaps, to provide minuends of different lengths. Then, too, we need not distribute the practice on the 100

primary facts evenly but we may well provide extra practice on those which are more difficult. To illustrate more fully the type of practice material which the teacher may use profitably, Set 2 has been prepared. In this set, minuends range in length from three digits to seven digits. Each of the 100 facts is included but the more difficult 50 of these have been included twice each. A few of the most difficult have been included three

Practice Examples in Subtraction. Set 2

Miscellaneous Examples

2784 1396	182 157	3504716 1247968	2475016 2130549
522674 345287	15267 9388	41003 35907	462528 274003
815148 764789	915 889	68444 24468	6417 1586
7301 5969	69244 39949	72884 16895	6265 5479
703 689	68312 42633	969233 705579	5830 4189
950745 761006	9035 236	1074 399	353 154
9521914 6850225	90226 41277	92473 13875	1678 899

times each. Thus, the practice exercises will contain examples of varying length and will emphasize the most difficult combinations.

In the two sets of practice examples which have been given for illustrative purposes, the number of digits in the subtrahend is usually the same as the number in the minuend. In no case does the number of digits in the two quantities differ by more than one. However, pupils should be given practice in subtracting small numbers from large numbers. Such examples as $\begin{array}{r}4627\\ \underline{93}\end{array}$, $\begin{array}{r}50000\\ \underline{29}\end{array}$, or $\begin{array}{r}149385\\ \underline{276}\end{array}$ need not be given frequently but they should occur occasionally in order that the pupils may feel confidence in their results when they solve examples of this kind.

HIGHER-DECADE SUBTRACTION

The role of higher-decade subtraction. In the discussions of addition, it was recommended that the pupils learn higher-decade addition soon after learning the fundamental addition combinations. It was pointed out that higher-decade addition is needed in column addition and also in carrying in multiplication. It is not uncommon to find teachers and other students of the teaching of arithmetic who recommend that after the pupils have learned the basic subtraction combinations they should proceed at once to the higher-decade subtraction combinations.

However, the case for higher-decade subtraction is quite different from that for higher-decade addition. In a large proportion of addition examples, we need higher-decade addition; indeed we cannot well get along

without it. But higher-decade subtraction is not ordinarily used in subtraction examples. The one and only important use for higher-decade subtraction combinations is found in short division. This use corresponds to that of higher-decade addition in carrying in multiplication.

Since it has been recommended that the first division taught be long division rather than short division, there is no justification for teaching higher-decade subtraction until after the pupils have learned to solve division examples having one-digit divisors by the long division form and are ready for short division as a short method.[19] There is, perhaps, a question as to the advisability of teaching short division at all, particularly to pupils who have average or less than average ability. This question is discussed briefly in the next chapter. If and when instruction in short division is given, it should be accompanied or preceded immediately by instruction in higher-decade subtraction.

Higher-decade subtraction in short division. The use of higher-decade subtraction in short division is easily seen by solving a division example by short division. In the accompanying example, one must subtract 18 from 23 and 48 from 52. It is seen that in a higher-decade subtraction combination, a two-digit number is subtracted from a two-digit number leaving a one-digit remainder. Now, if the quotient, 387, is multiplied by the divisor, 6, the two higher-decade addition combinations, 48 + 4 and 18 + 5 are used in carrying. Thus, we have in this division example two higher-decade subtraction combinations and in the

$$\begin{array}{r} 387 \\ 6\overline{)2322} \end{array}$$

[19] Morton, R. L., *op. cit.*, pp. 285-289.

corresponding multiplication example two higher-decade addition combinations. Stating these combinations again with answers, we have, $\begin{array}{r}23\\18\\\hline 5\end{array}$, $\begin{array}{r}52\\48\\\hline 4\end{array}$, $\begin{array}{r}48\\4\\\hline 52\end{array}$, and $\begin{array}{r}18\\5\\\hline 23\end{array}$. The relationship between the subtraction combinations and the addition combinations is easily seen.

From this relationship, one would infer that the total number of higher-decade subtraction combinations used in short division is the same as the total number of higher-decade addition combinations used in carrying in multiplication. This is precisely the case. The number of these combinations has been shown to be 175 (see Table 1, page 57). There are also 175 higher-decade subtraction combinations which are used in short division. We have seen that 60 of the 175 higher-decade addition combinations which are used in carrying in multiplication require bridging. We find also that 60 of the 175 higher-decade subtraction combinations which are used in short division have the minuend and the subtrahend in different decades. In these, an operation similar to bridging in a higher-decade addition combination is required. These 60 combinations are more difficult than are the remaining 115.

Table 2 shows in detail the higher-decade subtraction combinations which are required in short division. In the first column of the table are listed the 27 subtrahends which are products of multiplication combinations. These are the product numbers which are listed in the first column of Table 1. In the second column of Table 2 are listed the minuends which are used with

TABLE 2. HIGHER-DECADE SUBTRACTION COMBINATIONS USED IN SHORT DIVISION

Subtrahend	Minuends	Number of Combinations	Number in Which the Minuend and Subtrahend Are in the Same Decade	Number in Which the Minuend and Subtrahend Are in the Different Decades
10	11-12-13-14	4	4	0
12	13-14-15-16-17	5	5	0
14	15-16-17-18-19-20	6	5	1
15	16-17-18-19	4	4	0
16	17-18-19-20-21-22-23	7	3	4
18	19-20-21-22-23-24-25-26	8	1	7
20	21-22-23-24	4	4	0
21	22-23-24-25-26-27	6	6	0
24	25-26-27-28-29-30-31	7	5	2
25	26-27-28-29	4	4	0
27	28-29-30-31-32-33-34-35	8	2	6
28	29-30-31-32-33-34	6	1	5
30	31-32-33-34-35	5	5	0
32	33-34-35-36-37-38-39	7	7	0
35	36-37-38-39-40-41	6	4	2
36	37-38-39-40-41-42-43-44	8	3	5
40	41-42-43-44-45-46-47	7	7	0
42	43-44-45-46-47-48	6	6	0
45	46-47-48-49-50-51-52-53	8	4	4
48	49-50-51-52-53-54-55	7	1	6
49	50-51-52-53-54-55	6	0	6
54	55-56-57-58-59-60-61-62	8	5	3
56	57-58-59-60-61-62-63	7	3	4
63	64-65-66-67-68-69-70-71	8	6	2
64	65-66-67-68-69-70-71	7	5	2
72	73-74-75-76-77-78-79-80	8	7	1
81	82-83-84-85-86-87-88-89	8	8	0
	TOTALS	175	115	60

the subtrahends in the first column. These minuends are the numbers obtained by adding the carry numbers given in the second column of Table 1 to the product numbers in the first column. The number of combinations is shown in the third column and in the fourth and fifth columns is shown the number of combinations in which the minuend and the subtrahend are in the same decade and the number in which they are in different decades.

Teaching higher-decade subtraction. In teaching higher-decade subtraction, it seems best to emphasize its close relationship to higher-decade addition which has already been taught. We may begin with a brief review of those higher-decade addition combinations which are used in carrying in multiplication. Then the relationship of the subtraction combination to the addition combination may be pointed out and practice on combinations in the subtraction form may be provided. Soon, the related addition combination should withdraw to the background and direct attention should be given to the subtraction combination.

The pupil should discover at once that what he is learning is useful in solving division examples. After the pupil has become proficient in dividing by a one-digit number by long division, a few examples may be solved on the blackboard by short division that he may see the greater neatness, brevity, and ease with which the examples are solved when this method is used. Then he may see that he can use this newer and shorter division method if he learns higher-decade subtraction. Pupils vary greatly as to the extent to which short division appeals to them. Some are much interested and

attack the task of learning the higher-decade subtraction combinations eagerly. Others are entirely satisfied to continue using the long division process; and perhaps they should be.

Practice on higher-decade subtraction. If pupils are to use short division, they should develop a greater degree of proficiency in higher-decade subtraction than many pupils possess. We have already indicated that as these combinations are learned, their close relationship to higher-decade addition combinations should be appreciated. It is also important that the relationship of the higher-decade subtraction combinations to the basic subtraction facts be seen and appreciated. Thus, $\begin{array}{r}28\\ \underline{24}\end{array}$ is related to $\begin{array}{r}8\\ \underline{4}\end{array}$, $\begin{array}{r}52\\ \underline{48}\end{array}$ is related to $\begin{array}{r}12\\ \underline{8}\end{array}$, etc. Furthermore, the family relationships which were pointed out for higher-decade addition should also be seen in higher-decade subtraction. Thus, the combinations $\begin{array}{r}21\\ \underline{18}\end{array}$, $\begin{array}{r}31\\ \underline{28}\end{array}$, and $\begin{array}{r}51\\ \underline{48}\end{array}$ belong to a family and are closely related. Others in the same family can as easily be appreciated although they are not used in short division.

All of these relationships should be kept in mind in arranging early practice exercises on higher-decade subtraction. Also, the skills which the pupils are developing here should be put to use promptly in short division examples. It is well, at first, to construct these division examples so that only those higher-decade subtraction combinations having the minuend and the subtrahend in the same decade are required. Later, the more difficult combinations in which the minuend and the sub-

trahend are in different decades may be incorporated in division examples.

Eventually, it is well for drill and for review to duplicate a large number of these subtraction combinations on sheets of paper. We are giving here the entire 175 combinations in two sets of examples. For convenience, the 115 combinations whose minuends and subtrahends are in the same decade are given in Set 3 while the 60 combinations whose minuends and subtrahends are in different decades are included in Set 4.

Practice Examples in Subtraction. Set 3

Easier Higher-Decade Subtraction

34	28	25	17	14	41	46	66	76	88	78	59
32	25	21	16	12	40	45	63	72	81	72	54
11	89	47	37	29	57	68	17	73	84	38	74
10	81	40	32	25	54	64	15	72	81	35	72
69	18	59	45	82	33	27	16	65	44	19	31
64	14	56	42	81	32	21	12	64	40	18	30
42	85	38	22	15	43	64	13	27	24	35	47
40	81	36	20	14	40	63	10	24	21	30	45
21	44	83	49	65	32	16	19	43	12	69	58
20	42	81	48	63	30	14	16	42	10	63	56
36	17	66	75	39	29	17	48	26	23	24	36
35	14	64	72	35	24	12	45	25	21	20	32

37	29	28	48	26	14	86	55	68	33	45	57
36	27	24	42	24	10	81	54	63	30	40	56

16	39	67	25	18	28	35	47	87	22	38	27
15	32	63	24	15	27	32	42	81	21	32	25

19	77	56	23	67	34	49	29	13	46	18	39
14	72	54	20	64	30	45	28	12	40	16	36

79	37	19	58	15	46	26
72	35	15	54	12	42	21

Practice Examples in Subtraction. Set 4

Harder Higher-Decade Subtraction

26	31	40	50	62	23	80	71	35	55	44	20
18	24	35	49	54	18	72	63	27	49	36	14

40	52	34	20	63	32	41	52	62	34	51	20
36	48	28	16	56	28	36	49	56	27	48	18

22	61	53	30	50	71	53	25	32	21	60	41
16	56	48	24	48	64	45	18	27	18	54	35

61	54	21	31	50	22	43	54	70	51	30	33
54	48	16	28	45	18	36	49	63	45	27	28

23	42	70	33	53	30	24	51	31	52	60	55
16	36	64	27	49	28	18	49	27	45	56	48

Checking subtraction solutions. Checking in subtraction, as in addition is indispensable. The habit of check-

ing should be developed in the primary grades and maintained through the intermediate and upper grades.

The best check consists simply of adding the remainder to the subtrahend and comparing the sum with the minuend. In the primary grades, pupils frequently acquire the habit of writing the sum, as shown. This practice may persist in the intermediate grades. The practice may be justified for the first few examples which the pupils check in this manner in order that they may see the meaning of the process, but writing the sum should not become a habit. If teachers in the intermediate grades find pupils indulging in this practice, they should discourage it.

$$\begin{array}{r} 4876 \\ \underline{1928} \\ \underline{2948} \\ 4876 \end{array}$$

Solving problems. Skill in subtraction is not the important end in itself. Subtraction is learned for the purpose of enabling the pupils to solve problems involving this operation and to solve them with ease and accuracy. That this skill may not become a mere abstraction to the pupils, devoid of relationship to interesting and practical problem situations, many problems showing the uses of subtraction should be solved. These problems should be drawn largely from affairs with which they are concerned and which are known to be of interest to them.

QUESTIONS AND REVIEW EXERCISES

1. Find out just what instruction in the fundamentals of addition and subtraction is provided in the primary grades of your local school system. Does this program vary from teacher to teacher or from year to year?

2. Why is it insufficient for the teacher of a new group

of pupils to learn from a group test the speed and accuracy with which they can solve examples? What information should the teacher secure beyond that revealed through the use of the group test?

3. To what extent is remedial work in arithmetic a matter of individual teaching?

4. If a summary test on the addition facts is given in an intermediate grade, why should these facts be arranged in a miscellaneous order on the test sheet?

5. State and illustrate the difference between the terms, "addition combination" and "addition fact" as they are used in this chapter.

6. Are circuitous methods of response to be condemned when the pupil is learning the addition combinations? After the combinations presumably have been learned? How would you detect the use of such circuitous methods by pupils in an intermediate grade?

7. Are pupils who are slow in addition usually using wrong methods? Are such pupils always using wrong methods?

8. What is a higher-decade addition combination?

9. What are the two important uses of higher-decade addition? Which of these two uses is the more important?

10. Higher-decade addition combinations were divided into two major groups. What are they? Which of these groups is the more difficult?

11. Outline a plan for teaching higher-decade addition. How is higher-decade addition related to the basic addition facts? What is meant by the statement that the higher-decade addition combinations belong to families?

12. Why is it important in early work in column addition to provide systematically for practice on the basic addition facts as well as on the higher-decade combinations?

13. If the multiplier is 7, what are the possible carry num-

bers? What are the possible carry numbers which may be added to 32 in multiplication?

14. Does carrying in multiplication always mean higher-decade addition?

15. Does higher-decade addition involve carrying? If, in adding a column, one encounters the combination, $35 + 6$, how is he supposed to add 6 to 35?

16. Is it always true that telling is not teaching? Is it usually true?

17. Why is it important that pupils see the meaning of carrying? How would you teach pupils to understand carrying?

18. Why is our number system called a decimal system?

19. What is the duodecimal system of numbers? To what number in the duodecimal system is the number 17 in our decimal system equivalent? The number 30? 43? 75? 200? 500?

20. Why should the zero facts be omitted in early work with number facts? When should zeros be introduced in addition?

21. What is meant by the statement that zero is a place holder?

22. What precautions would you take in teaching pupils to add addends of unequal length?

23. By adding several long columns of numbers, find out what your own attention span is. Also determine the attention span of several other persons, including a few elementary school pupils.

24. What should the pupil learn to do when he reaches the limit of his attention span before reaching the end of a column?

25. What is the relative importance of speed and accuracy in addition? What is meant by the statement that speed is a symptom of mastery?

26. How should addition examples be checked? Why should such examples be checked? Should pupils in the intermediate grades be supplied with textbooks containing answers?

27. Which is the more complex, addition or subtraction? Why?

28. Practice each of the five subtraction methods until you are proficient in the use of each of them. Why should a teacher be proficient in the use of all of these methods?

29. Which method is the method of equal additions? The method of decomposition? The Austrian method?

30. What are the two major disputes as to subtraction methods?

31. Does one ordinarily subtract when he makes change? If not, what does he do?

32. Which subtraction method is recommended in the text? State the reasons for this recommendation?

33. How would you rationalize the operation of borrowing in subtraction?

34. What is higher-decade subtraction? How is it related to higher-decade addition?

35. What is the major use for higher-decade subtraction? Under what circumstances might it be desirable to omit higher-decade subtraction from the course of study?

36. How much higher-decade subtraction is needed if short division is taught? How can one tell which higher-decade subtraction combinations will be used?

37. Which higher-decade subtraction combinations correspond to those higher-decade addition combinations in which bridging is required?

38. What is the best way to motivate the learning of higher-decade subtraction?

39. How should subtraction examples be checked? In using this check, should any writing be done? Why?

40. Why is it important that practice on addition and subtraction be accompanied by the solution of problems in which these operations are employed?

CHAPTER TEST

For each of these statements, select the best answer. A key will be found on page 533.

1. The fundamentals of addition and subtraction are usually (1) begun in the primary grades (2) finished in the primary grades (3) begun in the intermediate grades.

2. Teachers usually (1) overestimate (2) underestimate (3) estimate correctly the difficulty of finding out what pupils have already learned.

3. Written tests usually reveal pupils' methods of work (1) adequately (2) inadequately (3) not at all.

4. The number of addition combinations without zeros is (1) 45 (2) 81 (3) 100.

5. The number of zero facts is (1) 45 (2) 10 (3) 19.

6. Pupils should think of $9 + 6$ as $10 + 5$ (1) when learning $9 + 6$ (2) always when they add 9 and 6 (3) not at all.

7. Slowness in addition means the use of roundabout methods (1) never (2) sometimes (3) usually.

8. The most important use of higher-decade addition is in (1) carrying in multiplication (2) short division (3) column addition.

9. Bridging is required in the combination (1) $\begin{array}{r}42\\ 6\\ \hline\end{array}$ (2) $\begin{array}{r}54\\ 4\\ \hline\end{array}$ (3) $\begin{array}{r}18\\ 2\\ \hline\end{array}$.

10. One of the combinations which is used in carrying in multiplication is (1) $\begin{array}{r}36\\ 8\\ \hline\end{array}$ (2) $\begin{array}{r}44\\ 2\\ \hline\end{array}$ (3) $\begin{array}{r}20\\ 5\\ \hline\end{array}$.

11. The total number of higher-decade addition com-

binations used in carrying in multiplication is (1) 45 (2) 100 (3) 175.

12. Combinations requiring bridging compared with combinations which do not require bridging are (1) easier (2) harder (3) equally difficult.

13. Telling is teaching (1) always (2) sometimes (3) never.

14. The number system should be learned (1) in the primary grades (2) in the intermediate grades (3) in both the primary and the intermediate grades.

15. Decimal numbers include (1) both whole numbers and fractions (2) whole numbers only (3) fractions only.

16. The decimal number, 90, when expressed as a duodecimal number becomes (1) 90 (2) 120 (3) 76.

17. The duodecimal number 100 when expressed as a decimal number becomes (1) 120 (2) 144 (3) 12.

18. Zero combinations should receive attention (1) after the others are taught (2) before the others are taught (3) while the others are being taught.

19. The symbol for zero has been used (1) longer than the other symbols (2) just as long as the other symbols (3) not so long as the other symbols.

20. Zero is a place holder (1) always (2) sometimes (3) never.

21. Pupils have trouble with zero combinations because (1) they are inherently difficult (2) they seldom occur in examples (3) they are poorly taught.

22. According to Courtis most children can add steadily for (1) 6 additions (2) 10 additions (3) 15 additions.

23. If a pupil reaches the limit of his attention span before reaching the end of a column, he should (1) hold the partial sum in mind while making a bodily movement (2) push on at once to the end of the column (3) begin adding the column again.

24. Compared with accuracy, speed is (1) more important (2) equally important (3) less important.

25. Answers were recommended for the intermediate grades (1) for all examples (2) for some examples (3) for no examples.

26. Compared with addition, subtraction is (1) less complex (2) equally complex (3) more complex.

27. The take-away-borrow method is also called the (1) method of decomposition (2) method of equal additions (3) Austrian method.

28. If the accompanying example is solved by the take-away-carry method, one thinks in the second step (1) 7 and 3 are 10 (2) 7 from 10, 3 (3) 6 from 9, 3.

$$\begin{array}{r} 7403 \\ \underline{2968} \end{array}$$

29. If the same example is solved by the addition-carry method, one thinks in the third step (1) 9 and 4 are 13 (2) 10 and 4 are 14 (3) 10 from 14, 4.

30. If the same example is solved by the addition-borrow method, one thinks in the fourth step (1) 2 and 4 are 6 (2) 4 and 2 are 6 (3) 3 and 4 are 7.

31. Solving this example by the complementary method, one thinks in the first step (1) 8 and 5 are 13 (2) 8 and 2 are 10 and 3 are 5 (3) 8 from 10, 2, and 3 are 5.

32. The most commonly used subtraction method is the (1) take-away-carry (2) take-away-borrow (3) addition-carry.

33. The major four subtraction methods should be known (1) by the pupils (2) by the teacher (3) by both pupils and teacher.

34. The take-away-borrow method was recommended because (1) it is the most widely used method (2) it is known to be the best method (3) it is the method which experts recommend.

35. The etymology of the word "subtraction" suggests

(1) a take-away method (2) an additive method (3) a complementary method.

36. In making change, one ordinarily (1) adds (2) subtracts (3) counts.

37. Attention span in subtraction compared with attention span in addition is (1) more important (2) less important (3) equally important.

38. Higher-decade subtraction is used in (1) short division (2) long division (3) carrying in multiplication.

39. Higher-decade subtraction compared with higher-decade addition is (1) more important (2) less important (3) equally important.

40. The best check for subtraction is (1) a repetition of the work (2) subtraction by a different method (3) addition.

SELECTED REFERENCES

1. Brueckner, Leo J. *Diagnostic and Remedial Teaching in Arithmetic.* Philadelphia: The John C. Winston Company, 1930, 341 pp. Approaches the study of arithmetic processes from the point of view of pupils' difficulties. Addition and subtraction of whole numbers are discussed on pages 110-135.

2. Buswell, G. T. and John, Lenore. *Diagnostic Studies in Arithmetic.* Chicago: The University of Chicago, 1926, 212 pp. This is a high-grade research study. Chapters II and III are especially valuable as a supplement for this chapter.

3. Dickey, John W. "Much Ado about Zero." *Elementary School Journal,* XXXII: 214-222, November, 1931. This article explains and illustrates inductively the use of zero in certain computations.

4. Grossnickle, Foster E. "Still More Ado about Zero." *Elementary School Journal,* XXXIII: 358-364, January,

1933. Comments upon the articles by Dickey and Wheat and suggests that the pupil has uses for zero other than that of a place holder.

5. Klapper, Paul. *The Teaching of Arithmetic.* New York: D. Appleton-Century Company, 1934, 525 pp. Chapters XX and XXI are devoted to the addition and subtraction of whole numbers.

6. Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company, 1937, 410 pp. Chapters 4, 5, and 6 deal with the teaching of addition and subtraction. The Selected References at the end of Chapter 6 (pages 211-212) include the more important sources of information on methods of subtraction.

7. Morton, R. L. "Higher-Decade Addition in Some Recent Drill Devices." *Journal of Educational Research,* XV: 104-110, February, 1927. Shows that prepared drill materials often provide a very poor distribution of practice on higher-decade addition.

8. Roantree, William F. and Taylor, Mary S. *An Arithmetic for Teachers.* New York: The Macmillan Company, 1932, 523 pp. Addition and subtraction are discussed in Chapter II as to teacher's knowledge and methods of teaching.

9. Taylor, E. H. *Arithmetic for Teacher-Training Classes.* New York: Henry Holt and Company, 1937, 432 pp. Addition and subtraction are treated in Chapters II and III.

10. Wheat, Harry G. "More Ado about Zero." *Elementary School Journal,* XXXII: 623-627, April, 1932. This article, inspired by the article by Dickey (Reference 3) comments upon the function of zero as a place holder and suggests that the zero combinations as such have no place in elementary arithmetic.

11. Wheat, Harry Grove. *The Psychology and Teaching*

of Arithmetic. Boston: D. C. Heath and Company, 1937, 591 pp. Considerable attention is given to the function of zero as a place holder in Chapters V, XII, and XIII. Chapters XIV and XV should also be read.

CHAPTER 3

THE FUNDAMENTAL OPERATIONS

MULTIPLICATION AND DIVISION

It has been suggested that the fundamentals of addition and subtraction be taught together. These two operations were seen to be closely related. Presenting the basic facts of the two operations as teaching units means economy in learning and a better understanding of the operations.

In like manner, the fundamentals of multiplication and division should be presented together. Thus, when we teach $\begin{array}{r}9\\ \underline{6}\\ 54\end{array}$ and $\begin{array}{r}6\\ \underline{9}\\ 54\end{array}$, we also teach $\begin{array}{r}9\\ 6\overline{)54}\end{array}$ and $\begin{array}{r}6\\ 9\overline{)54}\end{array}$. Again, four primary facts, two in multiplication and two in division, are presented as a teaching unit.

Entering fourth-grade pupils will have gone farther in their study of addition and subtraction than in their study of multiplication and division. Considerable variation is found in the amount of multiplication and division which is prescribed for the third grade but in no school with which the author is acquainted do the pupils go as far in multiplication and division as in addition and subtraction in this grade.

There is a current tendency to reduce the amount of multiplication and division which is prescribed for grade three. It has been a common practice to present the multiplication and the division combinations and also considerable elementary work in the solution of

examples with one-digit multipliers and one-digit divisors in this grade, but a few courses of study and a few textbooks now provide for only a beginning of these subjects in the third grade. They would carry ahead to grade four most of the work in multiplication and division which has been done in grade three. This may be a step in the right direction but what the ultimate practice will be can not be foretold at the present time.

THE TEACHING OF MULTIPLICATION

Taking an inventory. Since courses of study vary considerably as to the amount of multiplication and division to be taught in any grade and since pupils vary greatly in what they learn from early lessons in these subjects, the teacher's first problem is to ascertain precisely what each pupil knows about the operations of multiplication and division and what skills he already possesses. The teacher can not assume that a pupil is proficient in any phase of a subject because that phase is included in the course for the grade from which he has recently come. To begin a new topic for which the pupils are not yet prepared is to precipitate them into difficulties so serious that they readily become discouraged; if they learn at all, they learn by rote.

It is not unusual to find pupils trying to solve multiplication examples before they know the fundamental multiplication combinations. A fourth-grade girl whom the author observed, was trying to solve on the blackboard the example shown. After looking at the example for a moment, she made with her crayon four groups of marks, each group containing six marks as shown: |||||| ||||||

$$\begin{array}{r} 3576 \\ \underline{4} \end{array}$$

|||||| ||||||. Then she slowly and painstakingly counted the 24 strokes, wrote the 4 in the proper place, and recorded the 2 at one side, in cold-storage as it were, where she could find it when she needed it. She recognized, apparently, that it would be a long time before she would have further use for this 2. Then, of course, she erased the marks, made four groups of seven each in their place, looked over to the place where the 2 had been stored, and added two more marks to the row so that it looked like this: ||||||| ||||||| ||||||| ||||||| ||. The 30 marks were counted, the 0 was properly recorded, the 2 was erased to make a place for the 3, and the work continued. She consumed about 10 minutes in solving the example and, fortunately, obtained the correct answer.

Obviously this pupil knew the meaning of multiplication. Her actions showed a rather intelligent understanding of what was required. But she did not know the multiplication combinations. Apparently, she could not add either, or, she did not recognize that the product of 4 and 6 can be obtained by adding four 6's. Probably she found the sum of addition combinations by the mark-counting procedure also.

$$\begin{array}{r} 6 \\ 6 \\ 6 \\ \underline{6} \end{array}$$

While this pupil was busy at the blackboard, the teacher was occupied with other tasks. He gave the pupil no attention before she had finished. Then he inspected her answer and complimented her on the accuracy of her work!

Admittedly, this is an extreme case. But degrees of the same kind of faults are found in many schoolrooms. Sometimes, these faults are known to the teacher and

condoned by him but frequently the details of the pupil's methods of work are not known to the teacher who must plan his future work.

The fundamental combinations. There is a long and firmly established custom in this country of teaching multiplication combinations to twelve times twelve. This custom, which seems to be of English origin, has little to be said in its defense. In ordinary multiplication situations, one has no use for combinations beyond nine times nine. Of course, the tens will be learned while learning the nature and meaning of the number system, but this is not a matter of multiplication. A few twelves, perhaps as far as six twelves, should be learned eventually since there are 12 inches in a foot, 12 things in a dozen, and 12 months in a year, but they should be taught considerably later than the combinations to nine times nine. The elevens should not be taught at all. Probably, the time will come when the pupil will find it advantageous to learn a few fifteens since many articles are priced at 15 cents each, a dozen, a box, etc., but this time will probably be in the fifth or the sixth grade.

There are 81 primary multiplication facts if we omit the zeros and go to nine times nine but there are 144 if we go the twelve times twelve. The number of combinations in the former case is 45 and in the latter case, 78. Obviously, the task of learning 45 combinations is much simpler and easier than the task of learning 78. One important reason for the ineffective learning of multiplication combinations in many schools is the fact that energy and time are dissipated on the elevens and the twelves.

We have discussed in considerable detail the teaching of the multiplication combinations to nine times nine.[1] If these combinations have been moved ahead from the third-grade course to the fourth-grade course, the fourth-grade teacher should proceed in the light of the suggestions which have been given. But if the teacher in the fourth or a later grade receives pupils who have been taught these combinations, the first thing on the program should be a test to determine what the pupils know and can do. In Review Test 9, the 81 primary multiplication facts are arranged in a form which is convenient for testing purposes.

As was indicated in the preceding chapter, taking an inventory is largely a matter of getting acquainted with the habits and skills of the individual pupils. The information supplied by group tests is not sufficient. The teacher must know not only how accurate the pupils are but also what their habits of work are.

REVIEW TESTS FOR THE INTERMEDIATE GRADES. TEST 9

The 81 Multiplication Facts

8 6	7 3	9 4	6 2	7 1	9 9	4 5	9 1	4 2
1 6	7 8	3 3	5 5	4 6	2 3	1 2	8 7	6 5
8 4	5 2	2 8	1 3	7 9	3 5	1 7	2 6	5 3

[1] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company, 1927, Chapter 7.

6	9	8	5	3	7	8	1	2
4	5	8	6	8	2	1	4	5
2	5	4	6	3	5	7	7	9
4	1	8	7	4	9	7	5	3
1	3	2	6	2	4	7	4	8
1	2	9	3	2	1	6	5	9
3	2	1	2	3	4	4	3	6
9	7	8	1	6	4	9	7	1
5	9	5	3	8	7	6	9	4
8	6	7	1	2	4	6	8	3
1	8	9	4	6	1	8	9	6
9	3	2	7	9	5	5	7	8

It is often observed that some of these multiplication combinations are much easier than others. The ones, twos, and fives usually are included in the easier group. One might well say that multiplying by 1 is not really multiplying at all. Multiplying by 2 is just another way of expressing the doubles of the addition combinations. The fives occur so frequently and in so many situations that they are learned quickly after they are taken up in school if they are not known already. For testing and review purposes, then, we can devote our attention largely to those combinations which remain after the ones, twos, and fives have been eliminated. The number of combinations remaining is 21. These 21 multiplication combinations mean 36 multiplication

facts. The teacher in the intermediate grades should give special attention to these 36 multiplication facts. Most of the remedial work which is required will have to be done with them. They are listed for convenience in Review Test 10.

REVIEW TESTS FOR THE INTERMEDIATE GRADES. TEST 10

The 36 Harder Multiplication Facts

6	9	6	4	8	3	7	9	9
8	7	9	7	3	3	8	9	4
7	8	4	8	8	7	6	8	3
3	6	6	7	4	9	4	8	8
4	6	3	7	9	6	7	8	3
8	7	4	7	3	3	6	9	9
3	4	4	3	9	7	6	9	4
6	4	9	7	6	4	6	8	3

It is especially important that the results in Review Test 10 be analyzed with care. The teacher must know not only the items, if any, on which the class as a whole does poorly and the individual pupils who make the poorest showing but also the particular items which each pupil misses. A large sheet of paper should be ruled into 36 columns and as many rows as there are pupils in the class, with extra columns and rows for the names of the pupils, the multiplication facts, and totals. Then an "0" may be recorded for each item omitted and an "X" or the actual wrong product given,

for each item wrong. Thus a detailed record will be available for the items with which the individual pupils have difficulty. Totals at the right will indicate the pupils who are most in need of attention. Totals at the bottom of the sheet will indicate the facts which are most in need of further practice.

Multiplication without carrying. If multiplication without carrying has been taught in the third grade, there should be a review test made up of examples of this kind. If multiplication without carrying has not been taught in the third grade, this should be the next step in the fourth-grade program. Review Test 11 contains examples having two-digit multiplicands and one-digit multipliers but which do not require carrying. All of the 81 primary multiplication facts, except those in which the multiplier is 1, are included in this test. The use of 1 as a multiplier is deferred until two-digit multipliers are used since the pupils will never have occasion in a real situation to solve examples in which

REVIEW TESTS FOR THE INTERMEDIATE GRADES. TEST 11

Multiplication without Carrying

21 5	81 7	42 4	91 8	41 6	81 9	63 3	61 7	50 5	74 2
51 8	72 3	60 9	81 6	51 7	60 4	41 5	83 2	71 8	61 6
71 5	82 4	71 9	41 7	81 8	41 9	90 2	71 4	61 8	71 7

31	52	61	81	40	52	91	91	31	21
9	4	2	5	8	2	7	3	7	8
82	31	91	91	40	51	50	91	31	71
3	6	5	9	3	9	6	4	5	6
31	32	91	21	20	21	52	61		
8	4	6	9	7	6	3	5		

the figure 1 is used alone as a multiplier. This test also includes zeros in the multiplicand. Of course, the only zero facts in multiplication which are of practical value are those in which zero occurs in the multiplicand. Since there is no carrying in the examples of this test, and since the test is arranged so as to include all of the primary facts, except those indicated, it has been necessary to include some of the facts in which 1 or 2 occurs in the multiplicand more frequently than the others.

Other practice exercises involving multiplication without carrying are easily prepared. To provide systematically for the inclusion of all of the multiplication facts used in Test 11, a list of these facts should be prepared on a separate sheet of paper and checked off as they are included in examples. The examples may include those having three-digit multiplicands as well as those whose multiplicands are two-digit numbers.

Multiplication with carrying. The operation of carrying in multiplication should be rationalized by a method similar to that used for rationalizing carrying in addition.[2] Coins may be used for first lessons but eventually the pupils should understand carrying in

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[2] Morton, R. L., *op. cit.*, pp. 262-265.

terms of units, tens, etc., rather than in terms of the cents, dimes, etc.

Set 1 of the practice examples contains 40 examples each of which has a two-digit multiplicand and each of which requires carrying. The 72 facts in which 1 does not appear as a multiplier are practised in these examples. Eight of the more difficult facts are included twice each. Zeros are not used in these examples.

Practice Examples in Multiplication. Set 1

Multiplication with Carrying

26	39	47	16	53	25	19	34	37	95
4	8	5	7	6	3	2	9	3	4
42	26	46	98	37	18	57	14	28	67
6	2	8	5	4	9	7	4	8	9
38	94	19	27	38	36	17	25	49	98
7	3	6	9	2	5	8	5	7	3
47	67	16	59	89	57	78	19	57	82
2	6	3	9	4	2	6	5	8	7

Practice examples in multiplication, set 2, have three-digit multiplicands and one-digit multipliers. In this set, each of the 36 more difficult facts (see Review Test 10) is used twice. The set also contains 13 of the 24 combinations involving ones, twos, and fives—one fact for each of these 13 combinations—and five zero combinations. This set of examples not only provides valuable practice on multiplication with carrying but also gives an excellent summary of multiplication so far learned. This set may be used as a review test in the

fifth or the sixth grade. It will be noted that the carrying difficulty in Sets 1 and 2 requires the use of higher-decade addition, as indicated in Chapter 2.

PRACTICE EXAMPLES IN MULTIPLICATION. SET 2

Three-Digit Multiplicands

963 4	849 6	389 7	693 9	687 3	674 8
367 6	416 2	934 3	389 8	478 4	647 7
973 7	689 6	397 8	384 4	479 9	486 3
374 6	502 8	843 8	846 7	105 3	368 9
501 5	679 4	487 9	709 2	397 3	510 6

Further work with one-digit multipliers. If the fundamentals of multiplication have been well taught, the pupils should attain a greater and greater degree of skill as they practise on examples having one-digit multipliers. Multiplication should become gradually easier for them and, as a consequence, the speed with which they multiply should increase and their errors should become less numerous.

Also, multiplicands may be made longer. Instead of containing only three digits, they may contain four, five, or six digits. The teacher should remember, however, that in the practical problems of the every-day

world, large numbers in multiplication do not often occur. The longer the multiplicand, the less frequently it should be found in the examples given to the pupils for practice. The pupils should see that large numbers are multiplied in exactly the same manner as small numbers and they should have sufficient practice to enable them to solve with ease such examples when they are encountered.

In solving long examples, attention span may become an important factor. Pupils may hesitate and make errors because of attention span limitations rather than because the preceding work has not been mastered, although attention span difficulties do not seem to be so frequent in multiplication as in addition. The symptoms are similar to those described for addition in the last chapter. The treatment is also similar. The only thing which the pupil is in danger of losing at such a crisis is the number which he is carrying. He may retain this carry number by repeating it to himself just as he repeats the partial sum obtained in column addition.

Introducing two-digit multipliers. The first question to be answered in planning multiplication by two-digit numbers is whether the first multipliers used shall be those ending in zero or those ending in digits other than zero. Both practices are found in use in schools. The prevailing practice seems to favor the use of multipliers containing no zeros at first. This practice seems to be the better one to follow. Probably the use of multipliers like 20, 40, etc., will be understood better if they are postponed for awhile.

When pupils begin multiplying by two-digit numbers which contain no zeros, the principal new difficulty

has to do with the second partial product. Heretofore, the pupil has not multiplied by a number in any position other than units' place. Now, he finds that he must multiply by a 6 which is in tens' place. Furthermore, he must place the second partial product, 282, correctly. Eventually he should see that the 282 is placed where it is because it really means 2820 and that it means 2820 because the 6 of the 63 means 6 tens or 60. This explanation of why we place the second partial product where we do probably will not be understood by pupils who are just beginning to use two-digit multipliers. If we were to undertake to put this explanation across at this time, we should probably have to begin with two-digit multipliers ending in zero, and, even then, only the brighter pupils would be able to appreciate the reason involved.

$$\begin{array}{r} 47 \\ \underline{63} \\ 141 \\ \underline{282} \\ 2961 \end{array}$$

The first explanation of placing partial products may be less complete but just as satisfactory to pupils as the better and complete explanation which can come along a year or two later. Step by step, multiplication can be shown as follows:

$$\begin{array}{r} 47 \\ \underline{63} \\ \end{array} \qquad \begin{array}{r} 47 \\ \underline{63} \\ 1 \end{array} \qquad \begin{array}{r} 47 \\ \underline{63} \\ 141 \end{array} \qquad \begin{array}{r} 47 \\ \underline{63} \\ 141 \\ 2 \end{array} \qquad \begin{array}{r} 47 \\ \underline{63} \\ 141 \\ 282 \end{array} \qquad \begin{array}{r} 47 \\ \underline{63} \\ 141 \\ \underline{282} \\ 2961 \end{array}$$

When we multiply 7 by 3, we place the 1 of the 21 directly beneath the 3. Also, when we multiply 7 by 6, we place the 2 of the 42 directly beneath the 6.

It is well to have several examples solved on the

blackboard before the pupils try such examples alone in order that they may learn well the form of the solution. The first multiplication step is a familiar one; the pupils frequently have multiplied two-digit numbers by one-digit numbers. In this, they have learned to place the units figure of the product under the units figure of the multiplicand and the multiplier. Now, nothing could seem to be more natural than that they should also write the first figure of the second partial product under the figure which is being used as a multiplier. This is not a complete explanation but it will suffice for a time. It is satisfactory to the pupil and it is as much as he can understand at this stage of his progress.

After pupils have attained a greater degree of intellectual maturity, have become more highly proficient in multiplication, and have learned to multiply by two-digit numbers ending in zero, a more satisfactory explanation of the placing of the second and later partial products may be given. The multiplier may be broken up and the example written as follows:

$$\begin{array}{r}47\\ \underline{63}\end{array} = \begin{array}{r}47\\ \underline{3}\\ 141\end{array} + \begin{array}{r}47\\ \underline{60}\\ 2820\end{array} = \begin{array}{r}141\\ \underline{2820}\\ 2961\end{array}$$

Then the pupil may write his solution as shown at the right. But he readily sees that this is the same solution as that which he has been using except that the zero appears at the end of the second partial product. He now sees that this zero appears here because the 6 of the 63 means 6 tens, or 60. The zero can be omitted without affecting the product so the pupil continues to follow

$$\begin{array}{r}47\\ \underline{63}\\ 141\\ \underline{2820}\\ 2961\end{array}$$

the form of solution to which he is already accustomed. However, the solution now appears in a new light. It is more meaningful.

Naturally, the extent to which the writing of second and later partial products is rationalized will depend upon the intelligence of the pupils. Those who are much below the average intellectually may well omit much of this rationalization. This is one of the numerous respects in which the arithmetic program may be differentiated so as to adapt it to the varying intelligence levels found among pupils.

Constructing examples for practice. In constructing sets of practice examples having two-digit multipliers, all of the basic multiplication facts should be used. Special attention should be given to the more difficult facts. Zeros should be found in the multiplicands. The multipliers will include numbers in which 1 occurs. This is the first real opportunity for practice with 1 as a multiplier. This is easy when the pupils understand what is meant. The question, "How many is 1 five?" is no harder to answer than the question, "How many is 1 dog?" Just as the pupils have learned to think and say, "3 fives" and "3 twos," rather than "three *times* five" and "three *times* two," they now think "1 five" rather than "one times five." Although such forms of expression as "three times five" are useful and must eventually be known, the meaning of multiplication is much more readily grasped if the pupil, in solving an example, thinks "3 fives" rather than "three times five."

$$\begin{array}{r} 25 \\ \underline{31} \end{array}$$

The first multiplicands should be two-digit numbers. Later, three-digit and four digit numbers may be used.

In Set 3, 90 multiplication facts are used but 30 of the more difficult facts are used twice and a few, three times.

PRACTICE EXAMPLES IN MULTIPLICATION. SET 3

Two-Digit Multipliers

24	79	61	80	77	26	89	35
63	85	78	19	69	71	47	96
28	63	80	49	70	16	76	89
84	38	36	29	48	46	24	93
97	45	30	86	49	34	12	85
31	58	57	56	46	19	29	24
15	80	76	48	97	39	45	12
31	29	99	87	77	68	97	85

Set 4 is made up of 20 examples, each having a two-digit multiplier and a three-digit multiplicand. Again, 90 primary facts are all included while 30, chiefly the more difficult, are included twice. Thus, there are 120 multiplications in this short list of examples—six in each of 20 examples. It should be noted that in Set 4 the multiplication of 0 by a number is practised under better conditions than was the case in Set 3. In Set 3, 0 could occur only as the second digit of a two-digit number, or in examples such as $\begin{array}{r}80\\ \underline{19}\end{array}$. But one would ordinarily solve this example by multiplying 19 by 80 rather than by multiplying 80 by 19, for the former is briefer and, therefore, somewhat easier. In Set 4, however, 0 occurs as the second digit of a three-digit number. Here, one naturally thinks, "8 zeros, 5 zeros," etc.

Practice Examples in Multiplication. Set 4

Three-Digit Multiplicands

624	906	153	807	375
37	58	49	62	17
137	247	462	214	135
38	54	96	21	56
809	908	608	395	376
47	39	18	27	94
254	897	768	148	849
38	16	52	79	84

It should not be overlooked that the first example having a two-digit multiplier should be derived from a problem. Here is a typical problem. "Our school bought 35 paper tablets, each tablet containing 96 sheets. How many sheets of paper were bought altogether?" The development may proceed somewhat as follows:

Teacher: "How shall we find the number of sheets of paper in all 35 tablets?"

Pupil: "Multiply 96 by 35."

Teacher: "Let us write the example this way. Now multiply 96 by 5 as you have always multiplied. (The pupil does so.) You wrote the 0 of 30 directly underneath the 5. Now, let us multiply by 3. How many are 3 sixes?"

$$\begin{array}{r} 96 \\ \underline{35} \\ 480 \\ \underline{288} \\ 3360 \end{array}$$

Pupil: "18."

Teacher: "Since we wrote the first figure directly underneath the 5 when multiplying by 5, let us now write the 8 directly underneath the 3. Now finish multiplying by 3. (The pupil does so.) Now, we have multiplied 96

by the 5 and by the 3 of 35. We must add these results." The pupil adds and finds that the total number of sheets bought is 3360.

Zeros in the multiplier. The usual practice, in teaching pupils to multiply by a two-digit number the second of which is zero, is simply to tell them to "bring down" the zero and then to multiply by the other digit. There are two forms which commonly appear. They are:

$$\begin{array}{r} 426 \\ \underline{30} \\ 12780 \end{array} \qquad \text{and} \qquad \begin{array}{r} 426 \\ \underline{30} \\ 12780 \end{array}$$

The second form seems to be used more frequently than the first.

To the pupil, it seems quite right and proper that we should multiply 426 by 3 but the zero situation is rarely if ever understood. This is difficult to rationalize. Perhaps no rationalization at all should be attempted. Some like to start with 10 as a multiplier, deriving the example from some such problem as, "How many pieces of crayon in 10 boxes if there are 12 pieces in a box?" Then the answer is first found by addition, as shown. This seems quite natural, for the multiplication combinations originally were taught as cases of abbreviated addition.[8] After the answer is found to be 120, it is written beneath the multiplication example. Then the pupil sees that he will have the same result if he merely copies the 0 and multiplies 12 by 1. This is followed

$$\begin{array}{r} 12 \\ \underline{10} \\ 120 \end{array} \qquad \begin{array}{r} 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ \underline{12} \\ 120 \end{array}$$

[8] Morton, R. L., *op. cit.*, pp. 224-230.

by a variety of examples having two-digit multipliers ending in zero.

After considerable practice has been provided on examples having such multipliers, the pupils learn short methods for multiplying by 10, 100, 1000, etc., deriving the rules:

To multiply by 10, annex 0.
To multiply by 100, annex 00.
To multiply by 1000, annex 000.

After three-digit multipliers have been taken up, zero can occur in the second or middle position. In this case, it does not seem to be advisable to have the pupils do more than skip the zero and write the 9 directly underneath the 3 when multiplying by 3. A row of zeros as a second partial product can be written in but they probably are of little value in enabling the pupils to understand the example. Zero as a multiplier means little, if anything, to pupils in the intermediate grades.

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   623
   304
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  2492
1869
------
189392
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The fifth set of practice examples contains three-digit and four-digit multiplicands and two-digit and three-digit multipliers. In this set again, each of the 90 facts is included, some are included twice, and a few, three times.

Practice Examples in Multiplication. Set 5

Three-Digit Multipliers

4853	9627	8019	4765
276	385	94	149

3801	761	745	2138
501	627	308	34
308	6029	239	5401
78	206	917	538
457	896	697	769
469	654	357	98

Checking multiplication examples. What has been said about the importance of checking answers in addition and subtraction applies equally well to multiplication. When a pupil has finished the solution of an example, he should check it as a matter of course. He should not be given answers.

Multiplication examples may be checked by reversing the positions of the multiplicand and the multiplier. After solving the example shown, for instance, the pupil will then multiply the multiplier by the multiplicand and compare the two products. This check has the advantage of being simple and easily understood. In learning the multiplication combinations, the pupil has learned 3 eights and 8 threes in the same lesson and has observed the identity of the products. This check is similar and is one which the pupil can readily use.

28	43
43	28
84	344
112	86
1204	1204

This check is not so simple nor so easily applied, however, if the multiplicand has more digits than the multiplier. For instance, if 2475 has been multiplied by 36 and checked by multiplying 36 by 2475, as shown, the check has twice as many partial products as the

2475	36	4328	2000
36	2475	2000	4328
14850	180	8656000	16000
7425	252		4000
89100	144		6000
	72		8000
	89100		8656000

original solution and presents a greater number of opportunities to make mistakes in writing the partial products and in adding them. Still more striking is the difference between the original solution and the check of the second example shown. Surely, no one who had multiplied 4328 by 2000 would check by multiplying 2000 by 4328.

The best check for multiplication seems to be division. Ordinarily, the multiplication and the division facts are taught together and any type of example in multiplication is followed soon by the corresponding type in division. But when a new type of example in multiplication (such as one having a two-digit multiplier) has just been studied, the pupil is not yet acquainted with the corresponding type in division and, therefore, can not yet use division as a check. In his early work in multiplication, the pupil will have to check by following the simple expedient of repeating the work in the same way, or, interchanging multiplier and multiplicand, if this is feasible. Later, when the two processes have been taught, multiplication examples should be checked by dividing the product by the multiplier and comparing the resulting quotient with the multiplicand. Of course, there will be no remainder.

The check "casting out nines" is a valuable check

for use with long examples but it is hardly to be recommended for use in the intermediate grades. It may be introduced, perhaps, in the junior high school.

Individual diagnosis of work habits. When the work habits of individual pupils are studied in detail, a great number and variety of errors and habits of questionable value are sometimes found. Bruge studied the errors of a group of pupils in multiplication and found 68 errors or practices of doubtful value.[4] These errors and questionable habits were found by making use of the interview technique. Only in this manner can the actual mental processes of the pupil be revealed to the teacher. Bruge concluded that test papers do not reveal much that the teacher should know about the work habits of pupils. To learn what the thought processes of a pupil are, as he solves a multiplication example, the teacher must sit down with him and have him think aloud, as it were. A knowledge of these thought processes is absolutely necessary if the teacher is to provide the proper kind of remedial teaching.

The use of problems. Multiplication, like the other fundamental operations, is learned for the primary reason of enabling the pupils to solve problems which involve this operation. Pupils should see the usefulness of multiplication. As the various types of multiplication examples are taken up, the attack upon them should be motivated by problems which require their use. Further problems should be found as the work on examples gets under way.

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[4] Bruge, Lofton V. "Types of Errors and Questionable Habits of Work in Multiplication." *Elementary School Journal*, XXXIII: 185-194, November, 1932.

The best problems arise out of the activities in which the pupils are engaged. But these activities may not supply a sufficient number and variety of problems to make up the strong systematically planned program which pupils need and should follow. The resourceful teacher will derive other interesting problems from the affairs of other children and, to some extent, from the affairs of adults.

THE TEACHING OF DIVISION

Division, the most difficult operation. Division is more difficult than addition, subtraction, or multiplication. It is very difficult for many pupils. Considering his age at the time he studies it, division is probably the most difficult topic in the arithmetic of the elementary school for the typical pupil.

Thorndike has suggested three reasons for the difficulty of division.[5] The first reason has to do with the selection of the quotient figure. It is possible to give the pupil instructions which will make this somewhat easier but to make a good selection of a quotient figure usually requires more than tricks and devices; it requires judgment. This is particularly true when the divisor is a number of two or more digits.

Secondly, division is difficult because it is complex. One must select a quotient figure, he must multiply, he must compare his product with the partial dividend, he must subtract, he must compare his remainder with the divisor, and he must bring down a figure to make another partial dividend. No other operation has any-

[5] Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, p. 166.

thing like the complexity of division. Both subtraction and multiplication are involved in division. When the example is checked, addition is also involved if the divisor and quotient are numbers of two or more digits or if there is a remainder.

The third reason for the difficulty of division lies in the fact that it is hard to motivate. Pupils do not find as many uses for division as for addition, or subtraction, or multiplication. Some pupils like division for its own sake, but for many, its uses must be made real and attractive if they are to take an interest in it.

Good teaching will not make division easy but it will make it easier. The good teacher will search diligently for and will find worthwhile problems which require division. She will help the pupil to familiarize himself with the complexity of the subject and to acquire good habits. She will supply aids for estimating quotient figures and will lead the pupil gradually to use a better and better grade of judgment that will increase the likelihood that his estimates will be correct.

The good teacher will analyze the subject matter of division, will isolate the various difficulty elements, and will organize a series of lessons which, so far as possible, will introduce these elements one at a time. Too often, the pupil is discouraged at the outset by being precipitated into a maze of difficulties so strange and so serious that he is overwhelmed and lost. The good teacher will not hesitate to break with tradition by teaching long division before short division or by making other changes which will increase the pupil's chances of success.

Division in the primary grades. It has been the

practice in many schools to teach all of the division facts and to do considerable work with one-digit divisors in the third grade. This sometimes goes so far as to have in this grade examples with three-digit quotients and to have quotients containing zeros. There is a current tendency, however, to teach little if any division in the third grade. The program which has been outlined in Volume I includes more than is offered in the third grade in many schools.[6] Probably it contains more than should be offered in this grade.

It has been pointed out already that we would have the multiplication and the division facts taught together as teaching units. We would also teach pupils to solve a given type of division example soon after they have learned to solve a corresponding type of multiplication example.

Teach long division first. Until recent years, it has been the almost universal practice to teach children short division before long division. The argument in favor of teaching long division first has already been presented.[7] In brief, long division is much easier than short division for any given example. In dividing 5274 by 6 by short division, for example, the pupil must subtract an *invisible* 48 from a visible 52, must think, "How many 6's in 47?" when the 47 is only *partially visible*, must subtract an invisible 42 from a partially visible 47, etc. This is difficult, very difficult for beginners in division. But if the solution is written out in the customary long division form, there is less left to mental imagery, the

$$\begin{array}{r} 879 \\ 6\overline{)5274} \end{array}$$

[6] Morton, R. L., *op. cit.*, Chapters 7 and 9.

[7] *Ibid.*, pp. 285-289.

steps in the process are clearly portrayed, and the pupil will master the elements of a difficult process without the unfortunate condition of initial discouragement.

Division by one-digit numbers. In teaching pupils to divide by one-digit numbers, it is necessary that the lessons be planned so as to permit the pupils to move gradually from one difficulty level to another. The examples at each level should be pretty well mastered before the examples of the next higher level are undertaken. The major difficulty steps for examples having one-digit divisors are repeated here for convenience.

1. The primary division facts as $8\overline{)48}$.
2. Examples having two-digit and three-digit quotients, no carrying.
   (a) Divisor contained in first digit of dividend, as $3\overline{)69}$, $2\overline{)486}$.
   (b) Divisor contained in first two digits of dividend, as $3\overline{)126}$, $4\overline{)2484}$.
3. The primary facts, with remainders, as $2\overline{)17}$, $6\overline{)53}$.
4. Examples involving carry, no remainders.
   (a) Two-digit quotients, as $6\overline{)84}$, $4\overline{)172}$.
   (b) Three-digit quotients.
      (1) Carrying in first step only, as $5\overline{)3755}$, $7\overline{)3157}$.
      (2) Carrying in second step only, as $8\overline{)1696}$, $3\overline{)1575}$.
      (3) Carrying in both steps, as $4\overline{)2532}$, $9\overline{)3753}$.
5. Examples involving carrying, with remainders, as $6\overline{)86}$, $4\overline{)175}$, $5\overline{)3757}$, $8\overline{)1699}$, $9\overline{)3758}$.

6. Zeros in quotient, without and with remainders.

   (a) At end of quotient, as $4\overline{)40}$, $5\overline{)750}$, $3\overline{)32}$, $8\overline{)2643}$.

   (b) In midst of quotient, as $6\overline{)1206}$, $4\overline{)1612}$, $9\overline{)5419}$, $8\overline{)3218}$.

$$\begin{array}{r} 8 \\ 6\overline{)53} \\ \underline{48} \\ 5 \end{array}$$

The pupil's first acquaintance with the long division form will come with the examples of the third class in this outline. Examples of the second class represent a very slight step forward from the primary division facts, since there is no carrying in these examples. The pupil who learns to solve examples which represent the primary facts with remainders, is learning an important step which leads directly into division with carrying.

It will be noted that in classes 4 and 5, we present examples involving carrying, first without remainders and then with remainders. This does not mean that the pupil necessarily must work through all of the subdivisions of class 4 before an example is permitted to have a remainder. It means, rather, that when a new type of example is taken up, remainders may well be dispensed with for a time in order that attention may be focused on the new difficulty which the type represents. Later, the examples should be allowed to occur both without and with remainders, usually with remainders.

Review Test 12 may be a review test if the fundamentals of division have already been learned or a set of practice examples in the grade in which this type of division is first introduced. This test includes examples without carrying and without remainders. The test in-

cludes each of the primary division facts except those in which 1 is the divisor. There are no zeros in the quotients. Since there is no carrying, there are many repetitions of the quotient figure 1.

REVIEW TESTS FOR THE INTERMEDIATE GRADES. TEST 12

Division without Carrying

7)357	6)486	9)549	3)216	8)408	2)148
4)248	5)205	7)427	5)255	3)189	9)729
6)246	8)728	4)168	7)567	5)105	3)156
5)305	6)126	7)147	9)189	6)546	4)128
8)248	6)426	5)155	4)364	6)306	9)459
8)126	9)819	5)455	6)186	3)246	8)168
7)217	3)273	7)637	2)104	8)328	5)405
2)122	4)208	9)279	7)497	8)488	4)284
2)184	9)369	8)648	7)287	9)639	4)328
5)355	6)366	8)568	2)166		

Since this test provides a thoroughgoing review of the primary division facts, the results should be studied carefully in order that the individual division facts with which the various pupils have difficulty may be located. Of course, the pupils' habits of work should be observed also.

The process of carrying in division may be rationalized through the use of coins, as carrying in addition and multiplication and borrowing in subtraction were rationalized. Such a problem as the following may be used.

Alice and Kathryn earned 56 cents selling Christmas cards. If they divide the money equally, how much should each girl receive?

The pupil thinks of the 56 cents as 5 dimes and 6 cents. Dividing the 5 dimes by 2, he sees 2 dimes and 1 dime remaining. He thinks of the dime remaining as 10 cents, combines the 10 cents with the 6 cents, and divides 16 cents by 2. Then he sees that each girl should receive 2 dimes and 8 cents, or 28 cents. On the blackboard, he may see solution expressed as follows:

$$2\overline{)56\text{ cents}} = 2\overline{)5\text{ dimes and }6\text{ cents}} = 2\overline{)4\text{ dimes and }16\text{ cents}}$$

Beginning with the right-hand expression, he will write "2 dimes and 8 cents" as his quotient, using the short division form, of course. Then, coming to the middle expression, he will write "2 dimes" above "5 dimes," multiply, write "4 dimes" below "5 dimes" and subtract. Coming, finally, to the left-hand expression, he will employ the usual long division form.

The pupil should drop the reference to coins readily and see 56 as 5 tens plus 6 units, should divide 5 tens by 2, multiply, subtract, and divide 16 units by 2, noting that the 1 ten remainder can be thought of as 10 units and combined with the 6 units already there.

$$\begin{array}{r} 28 \\ 2\overline{)56} \\ \underline{4} \\ 16 \\ \underline{16} \end{array}$$

Set 1 of the practice examples contains 72 examples each having a one-digit divisor and a two-digit quotient. In this set, there are nine examples for each divisor from 2 to 9 inclusive. In each divisor group, the dividends are so chosen that each of the nine digits is used as the first quotient figure and that

those which can occur in the second place in examples with carrying are used. Of course, if there is carrying and the divisor is 2, the second quotient figure can not be 1, 2, 3, or 4; if the divisor is 3, the second quotient figure can not be 1 or 2; if the divisor is 4 or 5, the second quotient figure can not be 1. In this set, all possible remainders will be found for each divisor. It will probably not be desirable to use all of the examples of this set in a single practice exercise.

Practice Examples in Division. Set 1

Division with Carrying

3)206	7)325	5)172	2)31	6)224	4)312
8)217	4)139	9)716	3)74	6)334	7)135
9)315	5)284	6)114	2)72	3)238	8)412
7)197	3)171	9)832	4)50	5)391	6)279
4)387	5)473	8)306	2)96	7)389	3)40
7)261	9)415	8)395	5)60	6)169	8)128
2)119	4)225	9)732	2)52	2)152	9)515
3)140	4)93	6)389	8)679	9)223	5)116
4)180	6)584	9)125	3)106	5)228	7)574
2)131	7)453	2)196	8)501	6)328	5)335
4)271	2)175	7)641	9)618	4)358	8)760
3)250	5)447	8)590	6)493	3)289	7)517

The examples of Set 2 have three-digit quotients. In this set, there are four examples having 2 as the divisor and three examples with each of the remaining one-digit numbers (except 1) as divisor. It was necessary to include four examples with 2 as the divisor in order that all nine digits might appear in the quotient, for 1, 2, 3, and 4 can appear in the first place only, if there is to be carrying in each step of the example. Remainders appear in nearly all of these examples and the same remainder does not appear twice for any one divisor, except 2.

Practice Examples in Division. Set 2

Three-Digit Quotients

$3\overline{)2810}$	$8\overline{)3087}$	$5\overline{)933}$	$2\overline{)972}$	$6\overline{)5591}$
$4\overline{)1077}$	$9\overline{)2550}$	$2\overline{)719}$	$3\overline{)741}$	$7\overline{)4490}$
$9\overline{)6831}$	$6\overline{)1724}$	$4\overline{)734}$	$3\overline{)556}$	$8\overline{)1395}$
$6\overline{)2740}$	$4\overline{)2299}$	$2\overline{)357}$	$7\overline{)1986}$	$5\overline{)2852}$
$7\overline{)6705}$	$9\overline{)5776}$	$2\overline{)592}$	$5\overline{)1974}$	$8\overline{)2156}$

The examples of Set 3 and Set 4 have quotients containing zeros. In Set 3, the quotients are numbers of three digits, the last of which is 0. Practice is distributed over the remaining digits in divisors and quotients so as to give the larger amount of practice on the larger numbers.

Practice Examples in Division. Set 3

Zeros in Quotients

$6\overline{)4324}$	$8\overline{)5044}$	$5\overline{)4703}$	$9\overline{)7291}$	$7\overline{)5950}$

4) 3883	7) 1123	3) 1680	8) 3841	6) 5583
5) 3600	9) 5760	2) 1741	4) 1401	7) 6444
9) 3334	5) 901	8) 2003	3) 572	6) 2762

Each of the 40 examples of Set 4 has a three-digit quotient with a zero in the second place. The digits 2 to 9, inclusive, occur in the quotients nine times each and 1, eight times. The larger divisors occur more frequently than the smaller, 2 being used twice and 9, eight times.

Practice Examples in Division. Set 4

Zeros in Quotients

7) 2849	5) 4033	9) 2724	8) 876	6) 3617
9) 4557	4) 3231	7) 6308	6) 2434	3) 2124
8) 2422	9) 6316	5) 2049	4) 1220	7) 4261
9) 3686	7) 1424	8) 4866	6) 4209	9) 7240
6) 5419	8) 3261	2) 1004	5) 1505	3) 1216
4) 2410	7) 1424	9) 2763	7) 3546	5) 1036
8) 4011	9) 8137	4) 3605	7) 6358	6) 1838
9) 1856	3) 9029	8) 3217	2) 1417	5) 2507

This matter of zeros in the quotient, particularly in the second place of a three-digit quotient, is one of the most troublesome things in division. Pupils should be

correctly instructed and adequately practised in this difficulty as it occurs in examples having one-digit divisors before they learn to solve examples having divisors of two or more digits.[8]

Introducing two-digit divisors. It has been stated that division is the most difficult of the fundamental operations but that it can be made easier by careful planning and by good teaching. Careful planning means, among other things, thorough treatment of examples having one-digit divisors before more difficult examples are undertaken. Using the long division form for such examples and following the steps which were outlined on pages 130-131, pupils have an opportunity to fix well the primary facts, to become thoroughly acquainted with the complex routine of division (writing the quotient figure, multiplying, subtracting, bringing down), to learn to handle zeros in the quotient, to become accustomed to remainders, and to discover how solutions should be checked. By this time a sufficient number of worthwhile problems should have been found to establish the usefulness of division and the pupils should be confident of their ability to handle any ordinary division example having a one-digit divisor.

However, much remains for the pupil to learn. Examples having two-digit divisors involve difficulties not yet encountered. Since the pupils have learned the routine of division and have developed adequate degrees of skill in the use of the division and the multiplication facts and in subtraction, it is evident that the principal difficulty from now on will be the estimating

[8] Morton, R. L., *op. cit.*, pp. 306-314.

of quotient figures. In the example shown, for instance, it is hard for the novice to tell how many 38's there are in 248. Multiplying 38 by 6, subtracting 228 from 248, and bringing down the 3 should not be particularly difficult.

$$\begin{array}{r} 6 \\ 38\overline{)24839} \\ \underline{228} \\ 203 \end{array}$$

Classify divisors. It is easier to estimate quotient figures correctly with some two-digit divisors than with others. It seems best, then, to begin with those two-digit divisors which are easiest to use. That is, we should begin with those two-digit divisors with which the estimated quotient figure is most likely to be the true quotient figure.

The easiest two-digit divisors to use are those ending in 0 (10, 20, 30, etc.) for they represent but a slight modification of the type of divisors which the pupil already knows how to use. The next easiest are those just above the multiples of 10 (11, 12, 21, 22, 31, 32, etc.). Using the "apparent-quotient" method for these divisors, we find that the trial quotient is usually the true quotient. Next should come those two-digit divisors ending in 9 and 8 for they are just below the multiples of 10. Here we use the "increase-by-one" method and find that the trial quotient so obtained is usually the true quotient. Finally, we have the most difficult, those divisors ending in 3, 4, 5, 6, and 7.

Classify dividends. If the divisor is 10, 20, 30, etc. the trial quotient is always the true quotient. But if the divisor is 11, 21, 31, etc., or 19, 29, 39, etc., the trial quotient is sometimes not the true quotient. In the examples, $21\overline{)1445}$ and $39\overline{)1983}$, for instance, the trial

quotients 7 and 4 are incorrect. It seems best, then, when the pupil is getting his first experience with two-digit divisors, to select dividends for which the trial quotients will be the true quotients and to enable him to gain some proficiency in solving such examples before those whose trial quotients are incorrect are used.

Selecting divisors and dividends in this manner gives the pupil an opportunity to become acquainted with the general procedure of dividing by two-digit numbers before the more difficult cases are encountered. But, eventually, the pupil must learn how to estimate quotient figures with any divisors and how to test the correctness of his estimates. So, after a little preliminary practice in dividing by 10, 20, 30, etc., by 11, 21, 31, etc., and by 19, 29, 39, etc., the pupil should be given examples in which the estimated quotient figures are incorrect and he should learn how to test the quotient figures which he selects.

Zeros probably should not be permitted to appear in the quotient until other difficulties have been fairly well mastered. To be sure, the pupil has had zeros in the quotient while dealing with one-digit divisors but this difficulty, when it occurs with two-digit divisors, is not quite the same as when it occurs with divisors of one digit. Consider the two examples shown.

$$\begin{array}{r} 508 \\ 7\overline{)3561} \\ \underline{35} \\ 61 \\ \underline{56} \\ 5 \end{array} \qquad\qquad \begin{array}{r} 408 \\ 24\overline{)9813} \\ \underline{96} \\ 213 \\ \underline{192} \\ 21 \end{array}$$

The only way we can have a 0 in the second place in the quotient of the first example (or of any example having a one-digit divisor) is to have the first division come out even. But when the divisor is a two-digit number, we can have a 0 in the second place of the quotient even though there is a remainder from the first division.

The question of remainders. Since the pupils have become thoroughly accustomed to having remainders at the end of division examples, remainders should be allowed to occur with divisors of two or more digits about as they would by chance. In some textbooks and other practice materials, nearly all of the examples are arranged so that they come out even but in real life experience a division example rarely comes out even. Practice materials supplied to children should be freed from this artificiality, for the children come to look upon remainders with suspicion, to see in them an indication that an error has been made.

When the divisor is 2, the possible remainders, of course, are 0 and 1. These in the long run, will occur equally often. The example comes out even when the remainder is 0, so, when the divisor is 2, we should expect about one-half of our examples to come out even. When the divisor is 3, the possible remainders are 0, 1, and 2. Since we expect these remainders to occur equally often, we should expect about one-third of our examples to come out even when the divisor is 3. Likewise, when the divisor is 4, about one-fourth should have no remainders; when it is 12, about one-twelfth should have no remainders; when it is 40, about one-fortieth should come out even; etc.

Outline of difficulty steps. The following outline summarizes in brief space what has been suggested by preceding paragraphs. This outline represents in schematic form the work in division to be undertaken in the intermediate grades after the work suggested by the outline for one-digit divisors (pages 130-131) has been completed.

1. Two-digit divisors ending in 0, without zeros in quotients, without and with remainders.
2. Two-digit divisors ending in 1 or 2.
   (a) Trial quotients correct, no zeros in quotients, without and with remainders.
   (b) Trial quotients incorrect, no zeros in quotients, without and with remainders.
   (c) Trial quotients correct or incorrect, no zeros in quotients, without and with remainders.
3. Two-digit divisors ending in 9 or 8.
   (a) Trial quotients correct, no zeros in quotients, without and with remainders.
   (b) Trial quotients incorrect, no zeros in quotients, without and with remainders.
   (c) Trial quotients correct or incorrect, no zeros in quotients, without and with remainders.
4. Other two-digit divisors. Trial quotients correct or incorrect, no zeros in quotients, without and with remainders.
5. Any two-digit divisors, zeros in quotients.
   (a) Zero at end of quotient.
   (b) Zero in midst of quotient.
6. Three-digit divisors.

A greater number of classes of examples could have

been prepared showing in greater detail the difficulty steps involved in division with two-digit divisors. We have made the outline brief intentionally in order that the teacher might get well in mind the major difficulty steps.

Two-digit divisors ending in zero. Examples having two-digit divisors ending in zero represent the next forward step for the pupil who has learned to solve examples having one-digit divisors. At first, these examples are appreciably harder than the examples with one-digit divisors simply because the pupil is not accustomed to carrying the zero along. But after a time, the zero ceases to be a disturbing element and the pupil finds such examples as easy as others he has learned to solve.

To introduce this type of example, use a problem such as the following: "John saw his father pay $2.40 for gasoline which sold for 20 cents a gallon. How many gallons did John's father buy?"

TEACHER: "How shall we find the number of gallons of gasoline John's father bought?"

PUPIL: "Divide $2.40 by 20."

TEACHER: "$2.40 is how many cents?"

PUPIL: "240."

TEACHER: "This is the way we divide 240 by 20. How many 20's in 24. 1 is right. Write the 1 above the 4. Multiply 20 by 1. Write the 20 under the 24. Subtract. Bring down the 0. How many 20's in 40? If you can't tell, think, 'How many 2's in 4.' 2 is right. Write the 2 above the 0. Multiply 20 by 2. Write the 40 under 40. Then, John's father bought 12 gallons of gasoline."

$$\begin{array}{r} 12 \\ 20\overline{)240} \\ \underline{20} \\ 40 \\ \underline{40} \end{array}$$

Other illustrative examples will be solved. Some of

these will present more difficulty in estimating quotient figures. In the example shown, for instance, the pupil will determine the number of 70's in 532 and 420 by thinking, "How many 7's in 53?" and "How many 7's in 42?" This makes the example more like other examples which the pupil has already learned to solve. The zero in the divisor is simply carried along.

$$\begin{array}{r} 76 \\ 70\overline{)5320} \\ \underline{490} \\ 420 \\ \underline{420} \end{array}$$

Sets of practice examples may be prepared as sets have already been prepared for one-digit divisors. The sets which have been given in this chapter may be adapted to our present purposes by annexing a zero to each divisor and another digit to each dividend.

Two-digit divisors ending in 1 or 2. The pupil is now ready to undertake more difficult examples having two-digit divisors. We may begin with a problem.

Our principal brought us a package of 500 sheets of paper. There are 31 boys and girls in our room. If he divides the 500 sheets among us so that we each get the same number of sheets, how many sheets will each get? How many sheets will be left over?

TEACHER: "How shall we find out how many sheets there are for each pupil?"

PUPIL: "Divide 500 by 31."

TEACHER: "This is the way we divide 500 by 31. Think first, how many 31's in 50? If you can't tell, think, how many 3's in 5? 1 is right. Write the 1 above the 0 of 50. The rest of the example is just like those you have been doing. Write 31 under 50, subtract, and bring down the other 0. How do you tell how many 31's there are in 190?"

$$\begin{array}{r} 16 \\ 31\overline{)500} \\ \underline{31} \\ 190 \\ \underline{186} \\ 4 \end{array}$$

PUPIL: "Think, how many 3's in 19?"

TEACHER: "How many 3's are there in 19?"
PUPIL: "6."

TEACHER: "We shall try 6. Write the 6 above the second 0. Multiply 31 by 6. Write the 186 under the 190. Subtract. Then there are 16 sheets of paper for each boy and girl in the room and 4 sheets left over."

Set 5 of the practice examples contains examples having two-digit divisors ending in 1 or 2 and two-digit quotients. Each of the 18 two-digit numbers ending in 1 or 2 is included once in this set. The dividends are chosen so that the quotient figure obtained by using the first figure of the divisor as a trial divisor is in each case correct. Each of the digits 1 to 9, inclusive, occurs twice in the first place in the quotient and twice in the second place. There are remainders for 17 of the 18 examples in the set. The quotients contain no zeros.

PRACTICE EXAMPLES IN DIVISION. SET 5

Trial Quotients Correct

31)734	12)397	22)1791	61)4586
82)7973	52)2499	11)176	81)4792
41)2639	91)3571	72)5423	32)2988
62)1155	71)5790	21)1317	92)4324
51)1340	42)2291		

The same divisors which were used in Set 5 occur again in Set 6. In Set 6, however, more than one-half of the trial quotient figures are incorrect. In the first example, for instance, the use of 5 as a trial divisor gives

7 as a trial quotient; but the correct quotient figure is 6. In some of the examples of Set 6, the first trial quotient figure is incorrect, in some the second is incorrect, and in three both are incorrect. This set will help to bridge the gap from the easier, somewhat artificial examples of Set 5 to the more difficult examples which arise from a chance arrangement of digits in divisors and dividends. The quotient figures in Set 6 occur as did the quotient figures of Set 5.

Practice Examples in Division. Set 6

Trial Quotients Correct and Incorrect

$51\overline{)3528}$	$21\overline{)1223}$	$82\overline{)7207}$	$12\overline{)383}$
$92\overline{)3771}$	$71\overline{)992}$	$41\overline{)1923}$	$52\overline{)1298}$
$91\overline{)6002}$	$31\overline{)2113}$	$72\overline{)1418}$	$22\overline{)1623}$
$11\overline{)513}$	$62\overline{)1731}$	$32\overline{)1022}$	$81\overline{)7774}$
$42\overline{)3148}$	$61\overline{)3292}$		

The use of the examples of Set 6 will be preceded by definite instruction in the solution of examples of this kind. In this instruction, the attention of the pupils will be directed to the fact that sometimes the quotient figure obtained by using the first digit of the divisor as a trial divisor is correct and that sometimes it is incorrect.

This leads to an important point in the teaching of division. After recording the trial quotient figure and multiplying the divisor by it, the pupil must compare this product with the partial dividend (the number

formed by the dividend figures immediately above). If the product is larger than this number, a smaller quotient figure must be used. This is one of the comparison steps in division.

Of course, there is also another comparison step. In dealing with divisors whose second digits are large (if the increase-by-one method is used) the pupil sometimes finds that the remainder secured by subtracting the product from the partial dividend is as large as, or larger than, the divisor. Again, he must compare and assure himself that the remainder is smaller than the divisor.

The teacher will mention frequently this matter of comparison—comparison of the product with the partial dividend and comparison of the remainder with the divisor. If the absurd mistakes which are sometimes made are to be avoided, this matter of comparison must become a habit. It will become a habit if the teacher directs attention to it frequently and effectively.

Two-digit divisors ending in 9 or 8. It is suggested that the pupil use the increase-by-one method in dividing by a two-digit number if the second digit of that number is 9 or 8. This means, of course, that the first digit of the divisor will be increased by one to form the trial divisor. In the example shown, the pupil will estimate the first quotient figure by thinking, "How many 5's in 25?" instead of "How many 4's in 25?" The reason for the use of this method with such divisors is obvious; it yields the correct quotient figure much more frequently than does the apparent quotient method.

$$
\begin{array}{r}
53 \\
48\overline{)2576} \\
\underline{240} \\
176 \\
\underline{144} \\
32
\end{array}
$$

Grossnickle shows that if the apparent method is used for all examples having two-digit divisors (except those ending in zero), the correct quotient figure is obtained in 63.9 per cent of the cases, but that if the apparent method is used for divisors ending in 1, 2, 3, 4, or 5 and the increase-by-one method for divisors ending in 6, 7, 8, or 9, the correct quotient figure is obtained in 78.2 per cent of the cases.[9] In arriving at these figures Grossnickle included in the group for which the correct quotient figure is yielded by the apparent method some examples which other investigators do not include in this group. In dividing 256 by 28, for example, the trial quotient is 12 if the apparent method is used but since no quotient figure can be greater than 9, the pupil is instructed to try 9 in such cases. In this case, 9 is correct so this example is included in the group for which the correct quotient figure is obtained by the apparent method. The same disposition is made of all examples in which trial quotients of 10 or more are secured, if 9 is the correct quotient figure.

Before deciding whether we should use the increase-by-one method for two-digit divisors ending in 6, 7, 8, or 9, we should determine the proportion of such examples for which the true quotient figure is obtained by this method. The answer to this question has been worked out by Jeep and reported by Knight.[10] Of course,

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[9] Grossnickle, Foster E. "How To Estimate the Quotient Figure in Long Division." *Elementary School Journal,* XXXII: 299-306, December, 1931.

[10] Knight, F. B. "Some Considerations of Method." National Society for the Study of Education, *Twenty-Ninth Yearbook.* Bloomington, Illinois: Public School Publishing Company, 1930, pp. 143-267. The work of H. A. Jeep is reported on pages 162-167. See also: The National

the true quotient figure is determined in many such examples by the apparent method, but Jeep shows that there are 14,202 examples, or 35.4 per cent of all, in which the correct quotient figure is yielded by the increase-by-one method but not by the apparent method. Some of these 14,202 examples, however, have divisors ending in 2, 3, 4, or 5. Furthermore, if the increase-by-one method is used for all examples having two-digit divisors ending in 6, 7, 8, or 9, about 80 per cent of the trial quotients will be true quotients, but if the apparent method is used for these examples, only about 35 per cent of the trial quotients will be true quotients.

It seems best to have pupils use the apparent method for all two-digit divisors ending in 1, 2, 3, 4, or 5 and the increase-by-one method for those ending in 6, 7, 8, or 9. Naturally, the apparent method does not work so frequently for divisors ending in 3, 4, or 5 as for those ending in 1 or 2; nor does the increase-by-one method give correct quotient figures so frequently for divisors ending in 6 or 7 as for those ending in 8 or 9. The farther one goes in either direction from the multiples of 10, the less frequently the method in use works. Eventually, any method must be supplemented by good judgment, as we shall see.

Some writers object to the recommendation offered in the last paragraph because it means that the pupil must learn two methods for estimating quotients, one

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Council of Teachers of Mathematics, *Second Yearbook*. New York: Bureau of Publications, Teachers College, Columbia University, 1927, pp. 41-48. And, Department of Superintendence, National Education Association, *Third Yearbook*. Washington: National Education Association, 1926, pp. 208-213.

for use when the second figure of the divisor is small and the other for use when the second figure of the divisor is large. The criticism is not without point. Moreover, there are many examples for which the correct quotient figure can not be obtained by the use of either of the two methods given. But the use of two methods means much less of the trial and error type of procedure and, in the opinion of the author, is justified for this reason.

In Set 7 of the practice examples, the 18 two-digit numbers ending in 9 or 8 are used as divisors. Each dividend is selected so that the trial quotient is correct if the first digit of the divisor is increased by 1 to form the trial divisor. Again, each of the digits from 1 to 9, inclusive, occurs twice in the first place in the quotient and twice in the second place. Each example has a remainder.

Practice Examples in Division. Set 7

Trial Quotients Correct

$49\overline{)1828}$	$79\overline{)4923}$	$28\overline{)2674}$	$88\overline{)1690}$
$58\overline{)4322}$	$89\overline{)4325}$	$39\overline{)2017}$	$48\overline{)1271}$
$29\overline{)2429}$	$18\overline{)1200}$	$98\overline{)7010}$	$69\overline{)2562}$
$78\overline{)7345}$	$19\overline{)483}$	$68\overline{)5677}$	$99\overline{)4866}$
$59\overline{)3117}$	$38\overline{)715}$		

The 18 two-digit numbers ending in 9 or 8 are used again as divisors in Set 8, but in this set one or more

of the trial quotient figures in each example will be found to be incorrect. The suggestions already offered in connection with the examples of Set 6 apply to the examples of Set 8. Here again each of the digits from 1 to 9, inclusive, will be found twice in the first place in the quotient and twice in the second place.

Practice Examples in Division. Set 8

Trial Quotients Correct and Incorrect

$59\overline{)5386}$	$89\overline{)1337}$	$28\overline{)2412}$	$19\overline{)799}$
$78\overline{)4995}$	$48\overline{)1107}$	$39\overline{)2304}$	$98\overline{)7551}$
$18\overline{)688}$	$79\overline{)7193}$	$68\overline{)3538}$	$49\overline{)889}$
$99\overline{)7825}$	$29\overline{)728}$	$88\overline{)7306}$	$38\overline{)1798}$
$58\overline{)3717}$	$69\overline{)2484}$		

Other two-digit divisors. There are 45 two-digit divisors which have not yet been included in practice examples. They are the two-digit numbers ending 3, 4, 5, 6, or 7—nine of each. Each of these 45 numbers is used as a divisor in the examples of Set 9.

In making up the examples of Set 9, the dividends have not been chosen with particular reference to whether the trial quotients are correct or incorrect. We have seen that neither method of estimating quotient figures is as satisfactory with these divisors as with divisors ending in 1, 2, 8, or 9. This is especially true for divisors ending in 4, 5, or 6. In general, the pupil should use the apparent method for divisors ending in 3, 4, or

5 and the increase-by-one method for divisors ending in 6 or 7, especially in his first experiences with these divisors. But by this time, the pupil should begin to use some judgment in selecting quotient figures and, as he gains experience with these examples, the factor of judgment should become more and more prominent. The pupil must learn early to use his best judgment in selecting quotient figures but he must expect to make errors. It is important that he learn to be on the lookout for errors, to detect them, and to correct them. It is sometimes said that multiplication examples can be solved with a pencil but that division examples require both a pencil and an eraser. If the use of the eraser is gradually decreased, the teacher may be assured of the pupil's progress. In the example shown, the pupil should think: "6's in 24, 4 and no remainder; 7's in 24, 3 and a remainder of 3; 3 is probably right." Then he multiplies, notes that 192 is less than 243, subtracts, notes that 51 is less than 64, brings down the 9, and thinks: "6's in 51, 8 and a remainder of 3; 7's in 51, 7 and a remainder of 2; 8 may be right; I'll try 8."

$$
\begin{array}{r}
38 \\
64\overline{)2439} \\
\underline{192} \\
519 \\
\underline{512} \\
7
\end{array}
$$

Practice Examples in Division. Set 9

Other Two-Digit Divisors

$47\overline{)2930}$	$63\overline{)5028}$	$26\overline{)2216}$	$34\overline{)1664}$
$87\overline{)8400}$	$15\overline{)1189}$	$73\overline{)6320}$	$97\overline{)7111}$
$54\overline{)3211}$	$23\overline{)1850}$	$45\overline{)4237}$	$83\overline{)1494}$

$17\overline{)981}$	$75\overline{)2000}$	$56\overline{)1749}$	$37\overline{)2845}$
$94\overline{)4666}$	$67\overline{)2052}$	$35\overline{)2431}$	$13\overline{)717}$
$66\overline{)5441}$	$84\overline{)4095}$	$24\overline{)1987}$	$43\overline{)769}$
$95\overline{)5502}$	$53\overline{)4963}$	$77\overline{)5284}$	$36\overline{)1782}$
$86\overline{)6055}$	$64\overline{)3121}$	$27\overline{)2580}$	$44\overline{)2796}$
$74\overline{)2124}$	$93\overline{)6922}$	$16\overline{)822}$	$57\overline{)5518}$
$25\overline{)987}$	$96\overline{)6573}$	$55\overline{)4096}$	$76\overline{)6877}$
$33\overline{)1892}$	$46\overline{)3991}$	$85\overline{)5921}$	$14\overline{)1150}$
$65\overline{)3766}$			

Later, he may learn to multiply the divisor by the trial quotient without actually writing the product and to compare the product which he temporarily remembers with the partial dividend to see that it is actually smaller. Eventually, he may be able to subtract this invisible product from the visible partial dividend and to compare the remainder with the divisor without writing any figures. But whether or not he attains this degree of skill, we can expect him to grow in skill in estimating quotient figures until he reaches the point where he seldom finds it necessary to erase and correct his work. This degree of skill comes slowly, however, and to many it may not come at all. Its attainment requires much practice and no little ability on the part

of the pupil as well as careful teaching and unlimited patience on the part of the teacher.

In preparing the examples of Set 9, a record was kept of the frequency of the occurrence of each of the 10 digits (including 0) in the quotient. The larger numbers have been used more frequently than the smaller. 0 and 1 occur four times each; 2 and 3, six times, 4, eight times, 5, ten times; 6 and 7, twelve times; and 8 and 9, fourteen times.

The practice examples which have been provided for two-digit divisors have been constructed so as to give two-digit quotients. Of course, the same divisors will be used with dividends yielding quotients of three or more digits.

When examples yielding three-digit quotients have been taught, the troublesome cases of XOX quotients—quotients of three digits, the second of which is 0—should receive careful attention. In teaching pupils to solve such examples, they should be forcefully impressed with the fundamental principle that every time a figure is brought down from the dividend, something must be written in the quotient. If no other numeral can be written there, we place a 0 in the quotient.

The next two sets of practice examples, Sets 10 and 11, contain examples having two-digit divisors and three-digit quotients. In Set 10, no zeros occur in the quotients, except as the third quotient figure. In Set 11, the second figure is 0 in the majority of the cases. A few cases in which there are no zeros in the quotient are included as a means of keeping the pupils alert, of preventing their writing 0 as the second quotient figure as a matter of form.

The digits 0 to 9, inclusive, occur in the quotients of Set 10 with about the same relative frequency as was the case in Set 9. In the first and third quotient figures of Set 11, a similar distribution of practice will be found. In general, the larger digits occur about twice as frequently as the smaller.

Set 10 contains 36 examples. The divisors are selected so that there are four from each decade from which two-digit divisors come. Each of the digits 1 to 9, inclusive, occurs four times as the second digit of these divisors.

Practice Examples in Division. Set 10
Zeros in Quotients

48)13804	65)19272	81)25764	12)5517
27)12922	59)28914	73)41426	97)57162
32)31053	64)47121	93)80842	56)42802
18)15044	87)14733	21)5427	45)8838
76)28591	35)25507	54)37443	78)61563
14)10981	42)39376	89)75497	23)13612
39)26523	99)25478	68)26945	57)53024
13)6127	82)55939	75)45772	91)22864
24)13691	66)28337	31)14892	46)21421

Set 11 also contains 36 examples. Twenty-eight of these have quotients in which there is a zero in the second place. The divisors were selected as were those of Set 10, but no divisor which was used in Set 10 is used again in Set 11.

Practice Examples in Division. Set 11

Zeros in Quotients

33)13472	61)43252	28)25374	15)12021
44)31056	83)57281	52)16067	98)20965
71)39020	29)17622	45)31658	84)9197
34)20415	17)15372	96)26561	51)41187
74)59469	62)25383	85)47386	16)6552
67)40348	38)30474	77)53961	92)85246
22)19483	49)15068	53)47837	58)46791
37)33329	41)20809	95)62832	19)15244
79)32254	26)21308	86)77677	63)25678

Three-digit divisors. The suggestions which have been offered for teaching children to solve examples having two-digit divisors apply, in general, to examples having three-digit divisors also. The trial divisor will be the first digit of the divisor, or the first digit increased

by one if the second digit is large. The pupil should be encouraged to ignore the third digit of the divisor completely in estimating quotient figures. In solving the example shown, the pupil thinks, "3's in 17, 5 and a remainder of 2, so 5 is probably right." Later, he estimates the number of 328's in 1129 and in 1453 by noting the number of 3's in 11 and 14, respectively. Thus, the pupil learns that estimating quotients with a three-digit divisor is almost the same process which was used in estimating quotients in an example having a divisor of two digits.

$$\begin{array}{r} 534 \\ 328\overline{)175293} \\ \underline{1640} \\ 1129 \\ \underline{984} \\ 1453 \\ \underline{1312} \\ 141 \end{array}$$

In providing practice on examples whose divisors have three digits, we should take into consideration the various difficulties involved and plan definitely to give practice on each of them. Quotients will have two digits, three digits, and, sometimes, four digits. Zero difficulties will be included. All of the digits should appear in the quotients and the larger of these should be included somewhat more often than the smaller. Almost all of the examples should have remainders.

Practice Examples in Division, Set 12, have divisors of three digits. There are 36 examples in the set. Four of these have divisors in the one hundreds, four in the two hundreds, four in the three hundreds, etc. Each of the nine digits, 1 to 9 inclusive, is used four times as the second digit of these divisors and is not used twice in this position in the same hundreds group. The digits in unit's place in these divisors have been selected in the same manner. Thus, practice on three-digit divisors

is distributed so that pupils should be able to handle any three-digit divisors if they can handle these. Zeros have not been included in these divisors but they do not increase the difficulty of estimating quotient figures so much as they increase the difficulty of multiplication. Examples having zeros in divisors should be given in lists of practice examples also. They are easily prepared.

We have seen how instruction in division should proceed through a series of carefully graded steps from the easiest examples to the most difficult which the pupils are likely to encounter. We have suggested that, so far as practicable, but one difficulty should be introduced at a time and that practice should be provided on it until it is fairly well mastered before the next difficulty is introduced. We have suggested that practice examples be selected so as to include all of the basic division facts but that the more difficult of these be given the larger share of our attention.

Practice Examples in Division. Set 12

Three-Digit Divisors

$517\overline{)240731}$	$831\overline{)231472}$	$243\overline{)23465}$
$488\overline{)408670}$	$799\overline{)406923}$	$123\overline{)222831}$
$962\overline{)82657}$	$656\overline{)521378}$	$375\overline{)306217}$
$548\overline{)1649209}$	$214\overline{)167262}$	$826\overline{)63822}$
$661\overline{)626853}$	$392\overline{)266017}$	$757\overline{)446216}$
$433\overline{)21584}$	$185\overline{)1103938}$	$979\overline{)379594}$

788)588260	226)20666	599)361621
347)97164	613)526954	874)69364
935)288480	151)129045	462)319540
146)532845	554)403826	882)172361
677)188255	498)321691	913)325825
235)150753	761)32526	329)166061

The indiscriminate use of just any examples in division early in the pupil's experience with the subject is almost certain to precipitate him into difficulties from which he can extricate himself with great effort, if at all. It is of fundamental importance that the examples which pupils are asked to solve while division is being learned be examples which are adapted to their level of progress. Before examples from a textbook are assigned, they should be solved by the teacher in order that the difficulties which they contain may be known. This is particularly important for older textbooks or other textbooks which are not known to be prepared according to these standards.

THE USE OF SHORT DIVISION

Short division a short method. As its name implies, short division is a short method. Ordinarily, short methods are learned after the corresponding longer methods have been mastered. Short methods for multiplication, for example, are never learned until the

regular multiplication procedure has been learned. Pupils do not learn to multiply by 25 by annexing two zeros and dividing by 4 before they have learned to multiply by the usual method. For some reason, unknown to the author, division has very generally been taught by a short method before it was taught by a long method. A possible explanation is found in the fact that the examples having one-digit divisors usually were considered to be short division examples while examples having divisors of two or more digits were thought of as long division examples and, quite naturally, those in the former group were taught before those in the latter group. However, the distinction between long and short division is a distinction between two methods of solution, regardless of the number of digits in the divisor. The example, $4\overline{)136}$ is a short division example only if it is solved by short division. If the long form of solution is used, this example is an example in long division. Likewise, the example, $25\overline{)5625}$, is a long division example only when it is solved by long division. Many persons would solve this example by short division; to them, it is a short division example.

Short division more difficult. We have already indicated that short division is more difficult than long division, particularly for those who are just learning to divide. The reason is obvious. In short division, one must visualize numbers which are not written and make certain combinations of them with numbers which are in sight, while in long division the work is all written out where it can be seen. This matter of relative difficulty of the two procedures led the author in 1927

to recommend that long division be taught before short division.[11]

Should short division be taught at all? Many teachers have taken pride in the fact that their pupils never used long division for an example having a one-digit divisor. Others, after teaching short division before long division, later have accepted the work of pupils with apparent indifference, whether examples having one-digit divisors were solved by short division or long division. And, it may be observed, many pupils use long division for all examples after they have learned it, even though they have been taught short division first. Apparently, they find long division to be the easier process for all examples.

There is difference of opinion as to whether short division is sufficiently worthwhile to justify the time and energy required to learn it. John recommends that the long division form be taught first and that short division be taught later *as a short cut.*[12] Buckingham also recommends the later teaching of short division as a short method.[13] Ballard holds that there is only one

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[11] Morton, Robert Lee. *Teaching Arithmetic in the Primary Grades.* New York: Silver Burdett Company, 1927, pp. 167-169. Also, *Teaching Arithmetic in the Intermediate Grades.* New York: Silver Burdett Company, 1927, pp. 75 and 97-101.

[12] John, Lenore. "The Effect of Using the Long Division Form in Teaching Division by One-Digit Divisors." *Elementary School Journal,* XXX: 675-692, May, 1930.

[13] Buckingham, B. R. "The Training of Teachers of Arithmetic." National Society for the Study of Education, *The Twenty-Ninth Yearbook.* Bloomington, Illinois: Public School Publishing Company, 1930, pp. 376-377.

method and that that method is long division.[14] Grossnickle concluded that it is not worthwhile to teach the short division form at all.[15] The present writer has recommended that we make the long division form "the King's Highway," as Ballard expresses it[16] until the pupil has learned well the fundamentals of this difficult operation, but that later, perhaps in the junior high school, the pupil may learn the short method when the finishing touches are being put on his work in division.[17] However, those below the average in general intelligence and arithmetical ability may well omit short division entirely.

Olander and Sharp investigated the use of long and short division with the harder one-digit divisors by 1265 pupils in grades IV to XII, inclusive, in four school systems. They found that three-fourths of these pupils used the long form, one-fifth the short form, and one-twentieth both forms. There was little difference between the grade pupils and the high school pupils as to the form used. They recommend the use of the long form at first and the short form later as a short cut for the more able pupils.[18]

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[14] Ballard, Philip Boswood. *Teaching the Essentials of Arithmetic.* London: University of London Press, 1928, p. 170.

[15] Grossnickle, Foster E. "An Experiment with a One-Figure Divisor in Short and Long Division." *Elementary School Journal,* XXXIV: 496-514 and 590-599, March and April, 1934.

[16] Ballard, P. B., *op. cit.,* Chapter VI.

[17] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company, 1937, p. 289.

[18] Olander, Herbert T. and Sharp, E. Preston. "Long Division *versus* Short Division." *Journal of Educational Research,* XXVI: 6-11, September, 1932.

In some schools in which short division is taught first and is the required form for examples having one-digit divisors, the alleged advantages of the short division form are lost, to a considerable extent, because of the use of crutches. In solving the example, $7\overline{)5362}$, for instance, it is not unusual for the pupil to write a small 4 between the 3 and the 6. This makes the partially visible 46 into a fully visible 46 and is a step toward the long division form. Also, pupils sometimes facilitate subtraction by writing the product below the partial dividend (here 49 under 53) or write the partial dividend and the product at one side before subtracting.

There are certain advantages in short division which should not be lost if the procedure can be learned without an undue expenditure of time. In the first place, it is neat and concise; it makes a better appearance than does long division. Secondly, it requires less space, thereby effecting a slight economy in materials. In the third place, when one has become expert at this form, it can be done more quickly than can long division. Finally, in solving problems requiring two or more steps where division is one of the steps, the quotient can be written out at once without performing the operation on a separate sheet of paper.

However, these advantages are not sufficient to justify the labor and time required for pupils who are below the average in intelligence and arithmetical ability to learn this division form. It seems best, then, to postpone short division until the seventh grade and then to offer it as a short method to those who are more able than the average. So used, it may add interest to division at a

time when further practice is required and increased interest may be desirable.

Teaching short division. If and when the teaching of short division is undertaken, it must be preceded or accompanied by the teaching of higher-decade subtraction. It was shown in the last chapter that there are 175 higher-decade subtraction combinations which are used in short division. It was shown also that these higher-decade subtraction combinations are related to the basic subtraction facts and to the higher-decade addition combinations which are used in carrying in multiplication. It should not be very difficult, then, for the pupil to learn higher-decade subtraction, but to attempt short division without giving attention to higher-decade subtraction is to court unnecessary learning difficulties.

The divisor, 2, can be used with any dividends as soon as the pupil is familiar with the higher-decade subtraction combinations, $\begin{array}{c}11\\ \underline{10}\end{array}$, $\begin{array}{c}13\\ \underline{12}\end{array}$, $\begin{array}{c}15\\ \underline{14}\end{array}$, $\begin{array}{c}17\\ \underline{16}\end{array}$, and $\begin{array}{c}19\\ \underline{18}\end{array}$. These are quite easy since they are all in the same decade and the remainder is 1 in each case. Likewise, the divisor, 3, can be used as soon as the pupil knows the subtraction combinations, $\begin{array}{c}13\\ \underline{12}\end{array}$, $\begin{array}{c}14\\ \underline{12}\end{array}$, $\begin{array}{c}16\\ \underline{15}\end{array}$, $\begin{array}{c}17\\ \underline{15}\end{array}$, $\begin{array}{c}19\\ \underline{18}\end{array}$, $\begin{array}{c}20\\ \underline{18}\end{array}$, $\begin{array}{c}22\\ \underline{21}\end{array}$, $\begin{array}{c}23\\ \underline{21}\end{array}$, $\begin{array}{c}25\\ \underline{24}\end{array}$, $\begin{array}{c}26\\ \underline{24}\end{array}$, $\begin{array}{c}28\\ \underline{27}\end{array}$, and $\begin{array}{c}29\\ \underline{27}\end{array}$. These are a little more difficult. As the divisor becomes larger, the number of higher-decade subtraction combinations needed increases rapidly and they become more difficult.

If the pupils are to learn short division, they should be interested in it and have a desire to learn it. It seems best for the teacher to solve a few examples on the

blackboard, using the short division form, in order that the pupils may see the brevity and apparent ease of the process. An explanation may proceed somewhat as follows:

Teacher: "How many 4's in 13?"
Pupil: "Three."

Teacher: "We write the 3 above the 3 of the 13 and multiply 4 by 3 but we do not write the 12. We subtract the 12 from the 13 mentally and remember that we have a remainder of 1. Now think of the 1 and the 4, making 14. How many 4's in 14?"

$$\begin{array}{r} 337 \\ 4\overline{)1348} \end{array}$$

Pupil: "Three."
Teacher: "What is the remainder?"
Pupil: "Two."
Teacher: "Think of the 2 and the 8 as 28. Then, how many 4's in 28?"
Pupil: "Seven."

If the easier examples are solved first—examples having small divisors and quotients of two or three digits—the pupils should be able to solve such examples for themselves after seeing several demonstrations at the blackboard. However, some pupils seem to find it very difficult to visualize numbers. They depend upon having actual figures in sight. Much of this difficulty may be due to lack of training, but even with training some continue to find it difficult. Marked differences in the success of pupils in learning short division are to be expected. Those who find it very difficult should not be required to spend very much time on it.

For practice in short division, the sets of examples with one-digit divisors which have already been given

in this chapter may be used. They constitute a series graded according to difficulty.

Checking results in division. Division may be checked by reversing the divisor and the quotient. That is, after the division has been completed, the dividend may be divided by the quotient which has been obtained and there is fair assurance that the work was correct if the original divisor is obtained as the quotient and there is the same remainder. In the example shown, for instance, the quotient obtained from dividing 1758 by 26 is 67 and there is a remainder of 16. Then, if 1758 is divided by 67, the quotient is 26 and the remainder is 16.

$$\begin{array}{r} 67 \\ 26\overline{)1758} \\ \underline{156} \\ 198 \\ \underline{182} \\ 16 \end{array} \qquad \begin{array}{r} 26 \\ 67\overline{)1758} \\ \underline{134} \\ 418 \\ \underline{402} \\ 16 \end{array}$$

This check has the advantage of giving intensive practice on division. If division is to be learned, the more practice the pupil gets, the better, other things being equal. Its disadvantage lies in the fact that dividing the dividend by the quotient may involve a type of division which has not yet been learned. For example, if 2499 is divided by 52, the trial quotient is correct in both steps and the example is comparatively easy. But if 2499 is divided by the quotient, 48, the trial quotient in the first step is incorrect whether the apparent or the increase-by-one method is used. Then, too, the quotient may contain a greater number of digits than the divisor thus introducing a kind of example for which the pupil is not yet prepared.

The best check seems to be multiplication. The pupil will multiply the quotient by the divisor, add in the

remainder, and compare the result with the dividend. This check is easily understood, is easily applied, and provides an excellent review of multiplication with multipliers of two or more digits.

The check, "casting out nines," is a valuable one for use with long and difficult examples but is not appropriate for use in the intermediate grades. It may be introduced in the junior high school.

The use of division in problems. We have said that the scarcity of practical applications is one of the reasons for the difficulty of division. But pupils are likely to lose interest in the subject unless many interesting problems involving division can be found. Then, we must find such problems and they must cover the various types of examples which have been taught.

Of course, textbooks are one source. But no one textbook is an adequate source. The teacher should have at hand several good texts from which the best problems may be selected. Other problems will arise in the affairs of the school, of the children's homes, and of the neighborhood. An alert teacher will find these and will bring in a wide variety for use in the arithmetic class. Also she will encourage her pupils to look for problems which require division. Other problems may be "made up" and worded in such a way as to appeal to the imaginations of the pupils. They like to solve problems about hypothetical picnics, hikes, camping trips, etc., which are sometimes the experiences of boys and girls. Teachers who have made a serious effort to find good problems requiring the use of division have been gratified with the success which they have achieved.

Abstract and concrete numbers. A number of years

ago, arithmetic texts and arithmetic teachers made much of the distinction between abstract and concrete numbers. Pupils learned that abstract numbers were numbers which were not applied to any particular thing, numbers like 26, 3, 17½, etc. They learned that concrete numbers were numbers which were attached to specific objects or quantities, as 6 baskets, $425, 5 gallons, 10½ pages, etc. This distinction led to certain rules which were to be followed in performing the operations with numbers.

Rules for abstract and concrete numbers. In adding, the addends must all be abstract or all concrete; if concrete they must be of the same denomination. Likewise, in subtracting, both the minuend and subtrahend must be either concrete or abstract. Then, in addition, the sum is the same kind of number as the addends and in subtraction the difference or remainder is like the minuend and subtrahend.

In multiplication, the multiplicand may be either abstract or concrete and the product is always like the multiplicand. The multiplier must always be abstract. In division, the dividend may be either abstract or concrete. If the dividend is abstract, divisor, dividend and quotient are all abstract. If the dividend is concrete, the divisor may be either abstract or a concrete number like the dividend. If the divisor is abstract, the quotient is like the dividend. If the divisor is concrete and like the dividend, the quotient is abstract.

These rules were prepared for a good purpose. They were intended to keep young learners from going astray by combining unlike numbers in absurd and impossible ways. They would also enable one to label his answer

correctly when he got it. Unfortunately, however, they accomplished neither of these purposes for the young learners were usually unable to understand the rules. The situations to which the rules referred were easier to understand than were the rules. Accordingly, the brighter children did not need the rules and the duller were unable to profit by them.

Strictly speaking, the rules are correct. We cannot add horses and cows and call the sum either horses or cows or any other specific animal name, but in actual practice we do add horses and cows and call the result "live stock." We do add men, women and children and call the result "population." The advocate of the abstract-concrete-number distinction will say, "Yes, but before adding horses and cows, you actually thought of each as animals and you thought of the men, women, and children as persons before you added them. So, after all, your addends were alike and the sum was like the addends." Quite so. That is why we say that, strictly speaking, the rules are correct. But, in actual practice, we do not hesitate to write,

3,528	men
3,641	women
7,329	children
14,498	population of our city.

The words, "men," "women," and "children," are convenient notes which help us to identify the addends. We have no difficulty in adding apparently unlike numbers and labeling the sum if we understand the conditions of the problem from which these numbers came.

If we do not understand the problem, no rules about like addends will help us out of our predicament.

The real distinction. The important distinction, as Thorndike says, is between a *number* and a *quantity*. "The newer methods find this distinction between abstract and concrete numbers undesirable in teaching, and find much better ways of attaining the desired ends. The valuable distinction is between a *number* (or abstract number as the older methods would call it) and a *quantity*, or number of units of a certain sort. . . . What we add and subtract and multiply and divide with are numbers. If we wish to know what seven boxes, each containing 144 pieces of chalk, contain in all, we do not multiply pieces of chalk by boxes or pieces of chalk by 7. We multiply 144 by 7. We may keep mental or written notes of what quantities the numbers meant so as to be able to know what quantity the number obtained as product means."[19]

Application to multiplication and division. In multiplication and division, we remember that we are operating on numbers, not quantities, and we use as multiplier or divisor the number which it is more convenient to use as such even though doing so may mean a violation of the rules which have been cited. If one wishes to find the cost of 325 note books at 8 cents each, he multiplies 325 by 8, rather than 8 by 325, because the former is easier. One knows that the answer is $26.00 and not 2600 note books or 2600 of anything else because the problem calls for the cost of 325 note books. A little

$$\begin{array}{r} 325 \\ 8 \\ \hline \$26.00 \end{array}$$

[19] Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, p. 186.

later, when decimal fractions are better known, the multiplier may appear as .08 but it is not difficult to conclude that the answer is $26.00 rather than $2.60 or $260.00 even though the multiplier appears as 8. To be sure, the same result can be obtained by following the rule which the older methods advocated and making the work appear as shown, but such a solution requires more time and space and is more likely to be accompanied by errors. Furthermore, no one outside of the school does it that way and it would be folly to learn a method which people do not use unless there were some clear gain in doing so.

$$\begin{array}{r} 8 \\ \underline{325} \\ 40 \\ 16 \\ \underline{24} \\ \$26.00 \end{array}$$

In teaching the fundamental operations, the teacher will do well not to mention abstract and concrete numbers in the presence of the pupils. She will avoid absurdities by seeing to it that the pupils understand thoroughly what the problems mean and what they call for and by setting a good example in the language which she uses and in the forms of solution which she employs. She will not say that note books times cents gives dollars or that cents times note books gives dollars. Rather she will say that we can find the cost of the note books by multiplying the number of note books by the number of cents which each costs. To discuss abstract and concrete numbers will merely confuse the issue.

QUESTIONS AND REVIEW EXERCISES

1. What is the advantage of teaching the multiplication facts and the division facts of a teaching unit together?
2. Find out how much multiplication and division, if

any, is taught in the third grade in your local schools.

3. If you were a fifth-grade teacher, what would be the nature of the inventory in multiplication and division which you would take at the beginning of the school year? Why is it important to take an inventory?

4. Do the teachers in your local schools teach the multiplication combinations beyond nine times nine? What multiplication combinations beyond this should the pupil know eventually? When should these extras be learned?

5. How many multiplication facts are there from one times one to nine times nine? How many of these were classified in an easier group?

6. What limitations do group tests have for inventory purposes? What would you do in taking your inventory besides giving group tests?

7. Why does Review Test 11 have no large numbers in the units' place in the multiplicands?

8. Outline the procedure which you would employ in teaching carrying in multiplication.

9. Is attention span ever an important factor in multiplication?

10. What explanation, if any, would you give for placing the second partial product where we place it in multiplication?

11. Is it important that a set of practice examples in multiplication include all of the basic multiplication facts? Why?

12. Which way do you prefer to write a multiplier of two digits, the second of which is 0? Why?

13. What is the best check for multiplication when a new type of example has just been taken up? Should this check always be used? Why?

14. What is the interview technique in diagnosing the work habits of a pupil? What value does it have?

15. What reasons are given for the difficulty of division? Will good teaching make division easy?

16. Were you taught long division or short division first? Do you agree with the recommendation that long division be taught first? Why?

17. If it is agreed that long division should be taught first, should this form be used for examples having one-digit divisors and no carrying?

18. What limitations are placed upon the second quotient figure if there is to be carrying and the divisor is 2? If the divisor is 3? If the divisor is 4?

19. Why is it easier to estimate the quotient figure with some two-digit divisors than with others? Which two-digit divisors are most likely to be accompanied by correctly estimated quotient figures?

20. What proportion of division examples should have remainders? Examine one or more textbooks to see how frequently division examples have remainders.

21. Is it more difficult to use a two-digit divisor ending in 0 than to use a one-digit divisor?

22. Why is it suggested that two-digit divisors ending in 9 or 8 be used before those ending in 3 or 4?

23. What is the apparent method of estimating quotient figures? The increase-by-one method? With what divisors should each be used?

24. Do you believe that pupils will be better off if they use only one method of estimating quotient figures?

25. How can a pupil increase the accuracy of his estimates of quotient figures through the use of judgment? Should the process of estimating quotient figures be purely mechanical?

26. The divisors, 13, 14, 15, 16, 17, and 18, have been called the "demons" by Grossnickle. (See reference 4 in Selected References.) Try each of these divisors on several

dividends. Do you see why the term "demons" is a good name for them?

27. How should the quotient be estimated when three-digit divisors are used?

28. Summarize the argument for teaching the long division form for one-digit divisors.

29. Should short division be learned at all? If so, when and by whom?

30. Is higher-decade subtraction used in short division? In long division? Illustrate by an example the use of higher-decade subtraction in division.

31. What is the best check for use in division? What other checks can be used?

32. Why is it important that pupils find frequent use for multiplication and division in problems?

33. What is an abstract number? A concrete number? Would you teach these terms to pupils in the intermediate grades?

34. Summarize the old-time rules for the use of abstract and concrete numbers in the four fundamental operations.

35. What cognizance should the teacher take of abstract and concrete numbers in teaching multiplication and division?

CHAPTER TEST

Decide whether each statement is true or false. Check your decision by the key on page 533.

1. It was recommended that division facts be learned along with multiplication facts.

2. There is a current tendency to reduce the amount of multiplication and division which is prescribed for grade three.

3. A satisfactory inventory is provided by a written group test.

4. The multiplication combinations should be learned to twelve times twelve.

5. The number of multiplication combinations to nine times nine, without zeros, is 45.

6. The only zero facts in multiplication which are of practical value are those in which zero occurs in the multiplicand.

7. The method for rationalizing carrying in multiplication is similar to the method for rationalizing carrying in addition.

8. It was recommended that the first work with two-digit multipliers be with those ending in zero.

9. All pupils should have the same explanation of reasons for all processes in the fundamental operations.

10. In preparing sets of practice examples in multiplication, the teacher should use any numbers at random.

11. Pupils should learn to multiply by 100 by simply annexing two zeros.

12. The best check for multiplication is division.

13. The pupils can use the division check for all of their multiplication examples.

14. Each new phase of multiplication should be introduced by a problem.

15. The best problems arise out of activities in which the pupils are engaged.

16. To discover how individual pupils multiply, it is necessary to use the interview technique.

17. Division is more complex than any one of the other three fundamental operations.

18. Practical problems in division are as plentiful in the experiences of pupils as are practical problems in multiplication.

19. The long division form should be used in one-digit division examples which do not require carrying.

20. If the divisor is 3 and there is carrying, the second figure of a two-digit quotient can be 2.

21. Division is the most difficult operation.

22. Ordinarily, it is easier to divide by 54 than by 59.

23. Examples having zeros in the quotients are comparatively easy for children.

24. Ordinarily, when the divisor is 8, about one-half of the examples will have no remainders.

25. The easiest two-digit divisors to use are those ending in zero.

26. The increase-by-one method was recommended for two-digit divisors ending in 3 or 4.

27. There are two comparison steps in the solution of a division example.

28. Three-digit divisors are as much harder than two-digit divisors as two-digit divisors are harder than one-digit divisors.

29. Short division is more difficult than long division.

30. Short division should be taught to all pupils.

31. Long division requires the use of higher-decade subtraction.

32. Most children who learn short division in grade three use it later for all examples having one-digit divisors.

33. Crutches were recommended for use with short division.

34. If short division is taught, it should be taught after long division and as a short method.

35. If short division is taught, it should be taught to all pupils.

36. Dividing the dividend by the quotient and comparing the result with the divisor is the best check for division.

37. The rules for abstract and concrete numbers provided that the multiplicand must always be an abstract number.

38. Pupils in the intermediate grades should learn the difference between a concrete number and an abstract number.

39. The important distinction, as regards abstract and concrete numbers is the distinction between a number and a quantity.

40. The teacher should permit the pupils to use as multiplier the number which it is more convenient to use as multiplier.

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2. Bruge, Lofton V. "Types of Errors and Questionable Habits of Work in Multiplication." *Elementary School Journal*, XXXIII: 185-194, November, 1932. Reports a detailed analysis of errors made and questionable work habits followed by a group of pupils in multiplication.

3. Grossnickle, Foster E. "How to Estimate the Quotient in Long Division." *Elementary School Journal*, XXXII: 299-306, December 1931. Shows the frequency with which the correct quotient figure is obtained when the apparent and the increase-by-one methods are used.

4. Grossnickle, Foster E. "Classification of the Estimations in Two Methods of Finding the Quotient in Long Division." *Elementary School Journal*, XXXII: 595-604, April, 1932. Arrives at the conclusion that the apparent method is almost as accurate as the increase-by-one method if the increase-by-one method is used for divisors ending in 9 but these are counted as belonging with the apparent method. Refers to the division demons also and recommends that they be given special attention.

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9. Olander, Herbert T. and Sharp, E. Preston. "Long Division *versus* Short Division." *Journal of Educational Research,* XXVI: 6-11, September, 1932. Reports an investigation of the use of the long and the short division forms by pupils who had learned both forms.

10. Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, 260 pp. Some of the difficulties of division are treated on pages 166-173. The topic, "Abstract and Concrete Numbers," is discussed on pages 186-189.

CHAPTER 4

GETTING ACQUAINTED WITH FRACTIONS

Fractions in the primary grades. An elementary understanding of some of the more common fractions can be acquired by the pupil long before he undertakes seriously the study of fractions in the intermediate grades. The child in the primary grades has so many experiences which involve fractions that he learns much of the meaning of fractions in a very practical and concrete way.

A careful study of what young children know about fractions has been made by Polkinghorne.[1] She studied 266 children in the kindergarten, the first, the second, and the third grades of the Elementary School of the University of Chicago. It was discovered that their knowledge and understanding of fractions far exceeded what many persons would have expected. The test used contained 42 items. These items required the children to give the examiner one-half of 2 pencils, to draw a line half as long as another line, to tell whether a little stick of candy was one-half, one-third, or one-fourth as long as a big one, to draw a picture containing one-half as many apples as another which contained four apples, to color a glass so as to show that it was three-fourths full of orange juice, to color three-fourths of a group of marbles, to tell what is the same as three-halves, as four-thirds, as four-halves, as five-fourths, as seven-fifths, and in other ways to show their understanding of frac-

[1] Polkinghorne, Ada R. "Young Children and Fractions." *Childhood Education,* XI: 354-358, May, 1935.

tions. Every one of the pupils tested gave evidence that he knew something about fractions. None gave correct responses to all 42 test items but one gave correct responses to 41, two to 40, three to 38, etc., the average number of correct responses was 11. The average number of correct responses in the kindergarten was 4; in the first grade, 6; in the second grade, 12; and in the third grade, 18. They knew more about unit fractions[2] than other fractions and more about proper fractions than improper fractions or equivalent fractions.

The children studied by Polkinghorne may have been somewhat better selected than are the children in the corresponding grades in an average elementary school but even if some allowance is made for selection, it is obvious that an understanding of fractions is growing in the kindergarten and the primary grades. However, Polkinghorne discovered that there was great variation among the pupils tested and that some of them were unable to respond satisfactorily to very simple and elementary test items. This leads to an important conclusion, namely, that there is no basic minimum in the way of knowledge and appreciation of fractions which the fourth-grade teacher can assume her pupils to possess.

Taking an inventory in the intermediate grades. There are schools in which some of the operations with fractions are taught in an elementary way in the fourth grade, although these operations are usually taught in grade five. There is a current tendency to teach the more difficult work in fractions in the sixth grade. But because of their contacts with situations involving fraction ideas,

[2] A unit fraction is a fraction whose numerator is 1.

fourth-grade pupils are bound to grow in their understanding of fractions. If left to their own devices, they will also gain misunderstandings and erroneous ideas. Therefore, the fourth-grade teacher, should plan activities which will help the pupils to gain correct ideas about fractions. She should also fill in gaps in the experiences of those whose understanding of fractions is fragmentary and less complete than is that of others.

In any intermediate grade, it is necessary for the teacher to begin by taking an inventory of what knowledge of fractions the individual pupils already possess. Tests, such as those used by Polkinghorne, will reveal what the pupils already know and do not know and will bring to light misunderstandings and wrong ideas which have developed. The teacher may find that some of the pupils have a correct idea of what one-half means, for example, while others may think that one-half is merely *a part* or a *portion* and not necessarily *one of two equal parts*. It is fairly common to hear young children talk about the "biggest half" and the "littlest half."

To obtain a satisfactory inventory, the pupils must be examined individually. In the fifth and sixth grades, group tests will help the teacher to make a rough appraisal of the pupils' knowledge of and skill in fractions but much that the teacher needs to know can not be obtained through the use of group tests. Various questionable work habits and specific types of errors are brought to light only through individual examinations and the use of the interview technique. In the fourth grade, almost the entire inventory examination must be individual.

Developing an understanding of fractions. In any grade and in any subject, the teacher must begin where the pupils are rather than where she thinks they ought to be or where the course of study indicates they should be. To fail to begin where the pupils are is to undertake abstract work in the operations with fractions before the pupils have an adequate foundation in concrete experiences with fractions. If the teacher would build her lessons upon the meaning theory of instruction, she must see to it that the pupils *understand* fractions before they undertake to add, subtract, multiply, or divide them, or find common denominators, or reduce them to lower terms or to mixed numbers.

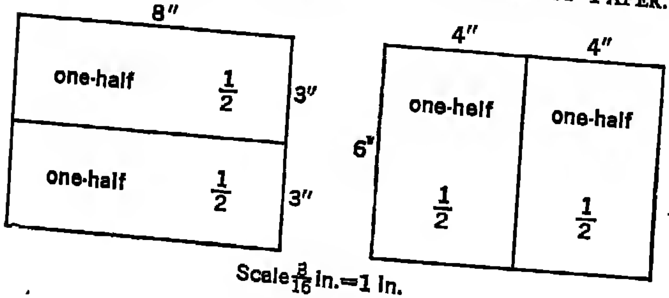
Some of the errors commonly observed in performing the operations with fractions are due to the fact that the pupils simply do not know what fractions are and what they mean. No pupil who has had adequate experience with fractions before undertaking to add them could obtain $2/6$ as the sum of $1/2$ and $1/4$, for example. If a pupil has a sufficient amount of the right kind of experience with $1/2$ and $1/4$, he will recognize at once that the sum is $3/4$, without any formalized procedure for finding common denominators, etc. A pupil, who has learned to think of $1/2$ and $1/4$ in terms of pies and dollars and other concrete materials certainly would never arrive at the conclusion that their sum is $2/6$.

Much of the so-called learning which takes place when fractions are studied in intermediate grades is nothing more than rote learning. Clearly, the theory of instruction upon which it is based is the drill theory. This means that the pupils do not understand what they do. Since they do not understand, they lack interest

in the subject. This lack of interest soon becomes an active dislike. No wonder tests given in the junior high school years and later reveal such a meager understanding of fractions.

We have indicated that the teacher should begin where the pupils are. Since the pupils vary so greatly in previous experience and attainment, much of the early instruction will have to be given to individuals or to small groups rather than to a whole grade class. The teacher may well begin in the fourth grade by making sure that everybody understands what is meant by one-half. A sheet of paper may be folded by the pupils, first longitudinally and then vertically. The crease will divide the sheet into *two parts* which will be *the same size*. When the pupils have had elementary work in finding areas of rectangles, they will see that the area of a smaller rectangle is one-half of the area of the larger rectangle no matter which way the paper is folded. If the sheet of paper is 8 inches long and 6 inches wide, the sheets will appear as shown in Figure 1.

FIGURE 1. TWO WAYS OF ILLUSTRATING THE FRACTION, ONE-HALF, BY FOLDING A RECTANGULAR SHEET OF PAPER.

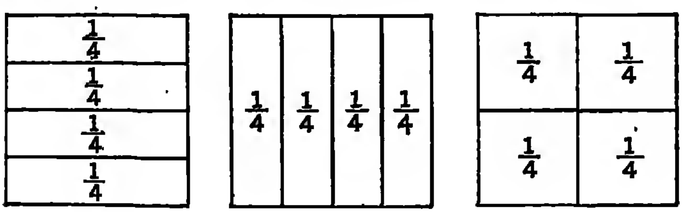


Other objective materials will be used to illustrate the fraction, one-half. At first these materials should be concrete materials, such as sheets of paper, apples, candy bars, etc. Then, semi-concrete materials may be used, such as rectangles and circles drawn on the blackboard or on paper.

In the primary grades, the pupil learns the meaning of one-half and learns to use the word "half," but he may not become acquainted with the abstract representation, $\frac{1}{2}$. As the subject is developed in grade four, the pupil learns to see as well as hear "one-half" and he also learns the symbolical form, $\frac{1}{2}$.

The fraction, one-fourth, should come next and should be represented in a similar manner. The sheet of paper may now be folded in three ways, as indicated by Figure 2. The pupil should see and understand that

FIGURE 2. THREE WAYS OF ILLUSTRATING THE FRACTION, ONE-FOURTH, BY FOLDING A RECTANGULAR SHEET OF PAPER.



if the sheet of paper is divided into four parts and these parts are all the same size, each of the parts is one-fourth and that it does not matter whether the sheet of paper is folded twice lengthwise, or twice crosswise, or once each way.

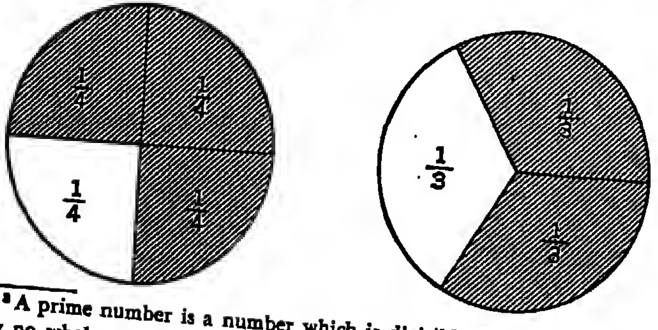
The teacher will observe that if the denominator of

the fraction is a prime number,[3] she can illustrate the fraction in the two ways shown in Figure 1 for $\frac{1}{2}$, and that there are only two ways if the rectangle is to be divided into smaller rectangles in the manner shown. But if the divisor is a composite number,[4] there are additional possibilities, as illustrated by the fraction, $\frac{1}{4}$, in Figure 2.

Other unit fractions to be illustrated in this manner with various objective materials will include $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{8}$.

In a similar manner, proper fractions which are not unit fractions will be illustrated concretely. The pupil sees that $\frac{3}{4}$ means 3 of the four equal parts into which something has been divided that $\frac{2}{3}$ refers to two of the three equal parts, etc. These two fractions are illustrated by Figure 3, in which a circle is used instead of a rectangle.

FIGURE 3. ILLUSTRATIONS OF THREE-FOURTHS AND TWO-THIRDS



[3] A prime number is a number which is divisible without remainder by no whole number except itself and 1, as 1, 2, 3, 5, 7, 11, 13, etc.

[4] A composite number is a number which is divisible without remainder by one or more whole numbers in addition to itself and 1, as 4, 6, 8, 9, 10, 12, 14, etc.

The pupils should have much experience with fractions, using concrete and semi-concrete materials, before undertaking to add or subtract them or to change their forms. Most pupils are not prepared to manipulate fractions until, through long experience, they have learned thoroughly what these fractions mean.

The illustrations of fractions which have been given have referred to parts of a single object or thing. The pupils may also apply the fractions which they learn to groups of objects. Thus, they may find one-half of four apples, one-third of six marbles, one-fourth of eight oranges, three-fourths of eight oranges, etc. This latter treatment should come later than the former for Polkinghorne found that children knew more about a unit fraction as applied to a single object than about a unit fraction applied to a group of objects.[5] Pupils also learn more easily that one object is one-half as large as another than they learn that one group of objects is one-half as many as another group of objects.

The fraction form for division facts. The division facts were found to have two meanings, *measurement* and *partition*.[6] When the pupil thinks: "how many 8's in 48?" as the reverse of "six 8's are 48," he is thinking in terms of the measurement idea of division. One measures 48 when he finds how many 8's there are in 48. The unit of measure is 8. But after learning that six 8's are 48, the pupil may also recognize that one of the six equal parts of 48 is 8. In other words, he sees

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[5] Polkinghorne, Ada R., *op. cit.*, p. 357.

[6] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades,* New York: Silver Burdett Company, 1937, pp. 282-285.

that *one-sixth* of 48 is 8. This is the partition idea of division.

All of the unit fractions from one-half to one-ninth, inclusive, may be used in this manner as an alternative way of expressing the division facts. Naturally, the pupil does not learn the terms *measurement* and *partition*, but the ideas which these terms represent become meaningful ideas to him. Having learned that multiplication is closely related to addition and having discovered the products of several multiplication facts by column addition, he obtains directly the answers to such division facts as $6\overline{)12}$, $5\overline{)15}$, and $3\overline{)12}$. Then, he learns also that one-sixth of 12 is 2, that one-fifth of 15 is 3, and that one-third of 12 is 4 and also that one-half of 12 is 6, that one-third of 15 is 5, and that one-fourth of 12 is 3.

		3
	5	3
6	5	3
6	5	3
12	15	12

The partition meaning of division is sometimes difficult to teach because the pupils have had inadequate prior experience with the required fraction forms. It is important, therefore, that the pupils have a rich and varied experience with these fractions before they undertake to use them as an alternative way to express the division facts.

Later, fractions other than unit fractions may be used in finding parts of numbers, as three-fourths of 8, two-thirds of 6, etc.

Work in division also leads to the use of a fraction as a part of a quotient. Thus, if 63 is divided by 4, a quotient of 15 with a remainder of 3 may be obtained, or the quotient may be written as 15¾. Some teachers

insist that all quotients which are not integers be written in mixed number form but whether a quotient should be written as a mixed number or as a whole number with a remainder depends upon the problem from which the division example has been derived. Thus, if a woman has 7 feet of toweling and wants to cut it so as to make two towels of equal length, it is clear that there will be 3½ feet of toweling for each towel. Here, 7 divided by 2 yields a mixed number quotient. But if 7 chairs are to be carried to another room by 3 children and each child can carry 2 chairs, it is equally clear that 1 chair will be left over. Under no circumstances, could one say that the number of pupils required would be 3½. In this case, we should need 4 pupils.

Fractions in varied situations. We have seen that the pupils learn about a fraction as a part of a whole or as a part of a group, that they find fractional parts of numbers in connection with their early experiences in division, and that their first experiences with unit fractions are followed later by experiences with proper fractions which are not unit fractions. All that they learn should be applied and applied frequently in a wide variety of situations. They learn that one-fourth and a quarter mean the same thing. Then the coin which is usually called a quarter takes on new meaning when it is seen that it has this name because it is one-fourth of a dollar. The pupils probably have learned to use such expressions as "a quarter after two" and "a quarter to six" in telling time without a clear understanding of why the word "quarter" is used. Now, they learn that "a quarter after two" means that a quarter

or one-fourth of an hour has passed since two o'clock and that "a quarter to six" means that one-fourth of an hour must elapse before six o'clock. They also observe that 15 minutes is one-fourth of an hour and learn that 15 is one-fourth of 60.

It is frequently observed that teachers fail to capitalize on many of the opportunities present for developing an acquaintance with, and an understanding of fractions. Then, when they find the subject prescribed by the course of study as a prominent part of the required work of the year or the semester, they undertake in a brief period of time to develop the pupil's understanding of a subject which requires more time than is then available. A pupil's understanding of fractions and their meaning grows slowly and gradually. "It is the rule rather than the exception" Thorndike says, "that the meaning of numbers is known incompletely and vaguely at first and is filled out and clarified by use."[7] Efforts to crowd the process, or to build skills upon an insecure and inadequate foundation of concrete, meaningful experiences are almost certain to fail. There is little learning and what there is, is likely to be rote learning.

As pupils become acquainted with facts concerning measures, there will be many opportunities for the use of fractions. They learn that a pint of milk makes two glasses or that a glass is one-half of a pint; that a quart of milk makes two pints or that a pint is one-half of a quart; and, later, that a quart makes four glasses or that a glass is one-fourth of a quart. They learn about

[7] Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, p. 108.

half-hours and quarter-hours, about half-dollars and quarter-dollars, about half-pounds and quarter-pounds, about half-inches and quarter-inches, about half-gallons and quarter-gallons, etc. They see the quart as a quarter or fourth of a gallon. All these things are learned as apparent by-products of the activities in which the pupils engage, such as telling time, measuring and weighing.

The use of measures also provides an opportunity to teach the meaning of fractions whose numerators are numbers larger than 1, and to make comparisons between fractions which will lead later to the more formal reduction to lowest terms and the changing of two or more fractions to fractions having a common denominator. Pupils learn that two quarters are the same as a half-dollar and that two quarts are the same as a half-gallon and, hence, that two-fourths equal one-half. Later, they will see this in the written form, $2/4 = 1/2$. It is to be noted that the equivalence of these two fractions is shown concretely and not by dividing the numerator and the denominator of $2/4$ by 2. The latter will come later, as the equivalent of the former.

The meaning of three-fourths and its shorthand form $3/4$, can also be learned from experiences with measures. We talk about three-fourths of a dollar, three-quarters of an hour, three-fourths of a gallon, etc. In like manner, two feet are seen to be two-thirds of a yard; two cents, two-fifths of a nickel; three nickels, three-fifths of a quarter; thirty cents, or three dimes, three-tenths of a dollar; five inches, five-twelfths of a foot; four days, four-sevenths of a week; etc.

The illustrative material which is used in building

up the fraction concept should be within the realm of the children's experiences. If the material is concrete, if it pertains to their activities at home, at play, and at school, and if it is properly handled by the teacher, it is probable that the children will find their experiences with fractions interesting and that they will grow steadily in their understanding of the subject.

Many questions which require answers in terms of fractions can be framed from the children's experiences with measures. The following, which include some of those already suggested, are typical of the rich and wide variety which is possible.

A pint is what part of a quart?

A glassful is what part of a pint?

A glassful is what part of a quart?

A quart is what part of a gallon?

What part of a gallon would two quarts be? Three quarts?

What part of a dollar is a quarter? Two quarters? Three quarters? (Develop equivalence of quarters and fourths.)

What part of a foot is an inch? Two inches? Three inches? Four inches? Five inches? Etc. (Lead to the discovery that $2/12 = 1/6$, $3/12 = 1/4$, $4/12 = 1/3$, $6/12 = 1/2$, etc.

What part of a nickel is a cent? Two cents? Three cents? Four cents?

A nickel is what part of a dime?

What part of a quarter is a nickel? Two nickels? Three nickels? Four nickels?

What part of a dollar is a dime? Two dimes? Etc.

(Lead to the discovery that $\frac{2}{10} = \frac{1}{5}$, $\frac{4}{10} = \frac{2}{5}$, $\frac{5}{10} = \frac{1}{2}$, etc.)

What part of a dozen oranges are three oranges? Four oranges? Etc.

What part of a week is a day? Two days? Etc.

What part of a school week is a day? Two days? Etc.

What part of a year is a month? Two months? Etc.

What part of a bushel is a peck? Two pecks? Three pecks?

The teacher should remember that the pupil must have many experiences with fractions and that these experiences must extend over a long period of time before the operations with fractions are undertaken. We have said that the pupil's understanding of fractions grows slowly. He passes from one stage to another as he grows in his understanding of the subject but at no stage does he know fully the meaning of fractions. If each experience and each stage find him with a better grasp of the subject, progress is being made, but we must expect progress to come slowly and gradually.

The use of terms and definitions. There persists in some schools a tendency to begin the study of fractions with a series of definitions. In other schools, definitions appear soon after the study of the subject is begun. In either case, the pupil is burdened with the necessity of memorizing wordy statements which mean little or nothing to him since he knows so little about the subject he can not understand the definitions.

The favorite words for definition are *fraction, numerator, denominator, proper,* and *improper*. To these are often added *integer* and *mixed number*. Now, all of these terms are terms which the pupil should know

eventually and should use but he will learn them more readily and more easily without the definitions than with them.

We have said that the pupil will learn the meaning of a few fractions in an elementary way before he is confronted with the symbolical representations, $\frac{1}{2}$, $\frac{1}{4}$, etc. He learns the words, "one-half," "one-fourth," etc., before he learns the word "fraction" and he learns the word "fraction" long before he learns to state in words what a fraction is. Then if the teacher chooses to ask him what a fraction is, it should be quite satisfactory if he answers, "$\frac{1}{2}$ and $\frac{1}{4}$ are fractions." It is enough at this stage if he can identify a fraction when he sees one. Also, it is enough if he at first merely associates the word "numerator" with the number above the line in a fraction and the word "denominator" with "the lower number." After the difference between a proper and an improper fraction has been recognized, he may think of a proper fraction as simply a fraction like $\frac{1}{2}$, $\frac{1}{8}$, $\frac{3}{4}$, $\frac{7}{8}$, etc., and of an improper fraction as a fraction like $\frac{2}{2}$, $\frac{4}{3}$, $\frac{6}{5}$, $\frac{4}{4}$, etc.

As the pupil progresses in his study of fractions, he should grow in ability to state in words what a fraction is, what a numerator is, what is meant by the denominator of a fraction, and the like. The evolution of his definition of fraction may proceed through such stages as, "a fraction is a part," "a fraction is a part of something," "a fraction is one of the parts that are the same size," "a fraction is one or more of the equal parts of an object or a group of objects," and "a fraction is one or more of the equal parts of a whole." He learns that the denominator tells the number of parts or the size of the

parts, that the numerator tells the number of parts that have been taken, that a proper fraction is one whose value is less than one, that an improper fraction is one whose value is equal to or greater than one, that an integer is a whole number, and that a mixed number consists of an integer and a fraction. But it is much more important that he be able to identify all of these and to use the terms correctly than that he be able to form rigorous definitions. The mere memorizing of ready-made definitions especially early in the study of fractions probably will hinder more than help growth and development.

It is sometimes said that there are two groups of pupils to whom a definition will be of little value: (1) those who are unacquainted with the thing defined for they can not understand the definition; and (2) those who are already acquainted with the thing defined for they do not need the definition. Probably this is an overstatement of the case but it contains more truth than fiction.

Common versus uncommon fractions. Common fractions are ordinarily distinguished from decimal fractions. Historically common fractions appeared before decimal fractions. With the advent of decimal fractions, common fractions were termed "vulgar" fractions as a convenient means of distinguishing them from decimal fractions. "Vulgar" meant that they were ordinary; vulgar fractions were fractions of frequent occurrence. Just as the common people were at one time referred to as the vulgar crowd to distinguish them from the aristocrats, vulgar fractions were the ordinary or the commonplace fractions.

In this country, fractions which are not decimal fractions are no longer called vulgar fractions; they are universally called common fractions. Then let us think of common fractions as belonging to two groups, those of frequent occurrence and those of infrequent occurrence. In other words, let us think of common fractions as opposed to uncommon fractions.

Many of the fractions found in textbooks, tests, and courses of study are quite uncommon. It sometimes happens that the fractions which receive the greatest emphasis are not common but very rare. Eighths are common but ninths are uncommon; yet the practice materials supplied to some pupils emphasize ninths as much as eighths. A set of practice examples may contain sevenths as often as thirds or ninths as often as fourths.

We do not know just how frequently the various fractions occur in ordinary affairs. Even though we possessed this information, we could not use it as a sole guide in determining the frequency with which these fractions should occur in the pupils' practice exercises. Pupils should have most of their practice on fractions of frequent occurrence but enough of the less frequently used fractions should be included to enable pupils to generalize on the procedures which they learn. Fractions which are of common occurrence should receive the greater emphasis because the skills demanded in practical affairs should be the skills trained for, because examples containing very uncommon fractions are often quite difficult, and because it is difficult to motivate work on fractions which rarely occur.

Wilson found in a study of the uses of arithmetic that

about 95.3 per cent of all the fractions used in problems collected from ordinary social and business practices were included in the list, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{1}{5}$, $\frac{2}{5}$, and $\frac{4}{5}$.[8] It will be noted that this brief list includes $\frac{1}{5}$, $\frac{2}{5}$, and $\frac{4}{5}$, but not $\frac{3}{5}$. It also includes $\frac{1}{8}$ and $\frac{3}{8}$ but not $\frac{5}{8}$ and $\frac{7}{8}$. If these three fractions are added to the list, 96.1 per cent of all the fractions found are included.

These data indicate clearly that fractions in common use are usually those having small denominators and that halves, thirds, fourths, fifths, and eighths include most of the fractions which one ordinarily encounters. Obviously, the school program should emphasize these "common" fractions. But the teacher will not confine her teaching to those listed. She should include sixths, tenths, twelfths, and sixteenths, letting them come along later than the others, however, and giving less practice with them. If the sixths, tenths, twelfths, and sixteenths are included in the list taken from Wilson's study, 98.8 per cent of all the fractions which he found will be used in instructional materials.

It has been suggested that a sufficiently wide variety of fractions should be included to permit the pupil to generalize as to the operations which he performs. Although the pupil may never have a real need for adding thirds and sevenths, he may well solve one or two examples involving these fractions merely to observe that the procedure is the same as that which he has used in solving other examples and to demonstrate to himself that he can solve such examples. Ninths will also receive occasional attention. Of course, sevenths and ninths

[8] Wilson, Guy Mitchell. *What Arithmetic Shall We Teach?* Boston: Houghton Mifflin Company; 1926, pp. 23-29.

will be used with the division facts as has already been indicated. Other fractions which may come in for limited attention now and then include fifteenths, twenty-fourths, and thirty-seconds. If sevenths, ninths, fifteenths, twenty-fourths and thirty-seconds are added to the list taken from Wilson's study, it is found that we have used 99.6 per cent of all the fractions which he found.

The well-taught pupil will have learned eventually to convert unusual fractions to decimals when he encounters them. Then, if they must be added, subtracted, multiplied or divided, the work is easier and is much more meaningful. Many of the fractions already named should not be permitted to remain in common fraction form after the pupil has studied both common and decimal fractions. Such a common fraction as $^{13}/_{17}$ is an arithmetical monstrosity; it should not be tolerated in this form by a well-taught pupil. These unusual forms were quite common in the arithmetics of a generation or two ago. Smith found the following exercises in fractions in a textbook which was in common use early in the present century.[9]

1. Reduce $^{7}/_{11}$, $^{9}/_{13}$, $^{17}/_{33}$, and $^{11}/_{39}$ to similar fractions having their least common denominator.

2. Add $171^{3}/_{23}$, $161^{7}/_{46}$, and $23^{4}/_{5}$.

3. Find the combined weights of five men whose respective weights are $157^{3}/_{4}$ lb., $161^{5}/_{8}$ lb., $159^{1}/_{2}$ lb., $173^{5}/_{16}$ lb., and $169^{5}/_{8}$ lb. (The knowing of weights to a single ounce seemed to the author of this book to be perfectly natural.)

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[9] Smith, David Eugene. *The Progress of Arithmetic in the Last Quarter of a Century*. Boston: Ginn and Company, 1923, p. 9.

4. A man walked $23\frac{2}{3}$ mi. the first day of a trip, $25\frac{3}{20}$ mi. the second, $28\frac{14}{64}$ mi. the third, and $26\frac{53}{100}$ mi. the fourth. How far did he walk in all?

5. Find the value of $8\frac{3}{7} + 5\frac{4}{9} + 9\frac{2}{3} - 3\frac{8}{21} - 3\frac{6}{7}$.

6. Find the value of $2\frac{3}{49} \times 7\frac{3}{4} \times \frac{9}{10}$.

7. Find the value of $\dfrac{\frac{9}{19} \text{ of } 13\frac{7}{12}}{1\frac{8}{38} \text{ of } 7\frac{5}{16}}$

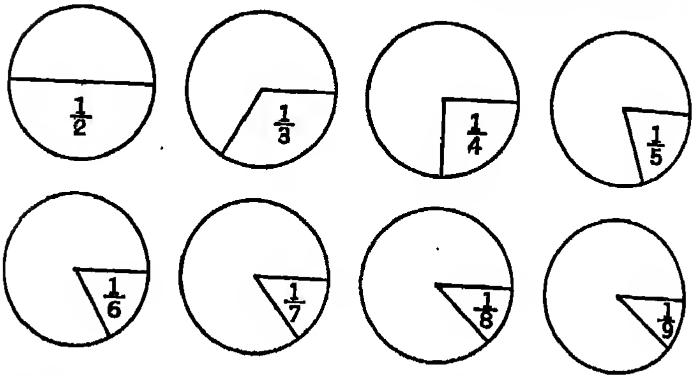
Pupils not only find their work to be less difficult but also more pleasant and interesting if the examples and the problems which they are assigned contain only fractions of reasonably frequent occurrence. One textbook, published much later than that from which Smith made his collection, requires in a single lesson that the pupils shall subtract sevenths from fifths, ninths from eighths, eighths from sevenths, ninths from fifths, sevenths from thirds, sevenths from ninths, elevenths from halves, halves from ninths, halves from elevenths, and thirds from seventeenths.

Comparison of fractions. When fractions are objectively represented, pupils can readily compare them and tell which is larger and which is smaller. If they are confined largely to the numerical form of expression, pupils quite naturally make errors, concluding that $\frac{5}{8}$, for example, is larger than $\frac{3}{4}$, simply because larger numbers are used as numerator and denominator. But if they take two sheets of paper of the same size, divide one into eighths and the other into fourths by folding or by ruling, they learn in a way which is more likely to be meaningful that $\frac{5}{8}$ is smaller than $\frac{3}{4}$. Indeed, they see that $\frac{3}{4}$ is exactly equivalent to $\frac{6}{8}$.

Comparison may begin with unit fractions. Any con-

venient diagram may be used. The favorites are circles and rectangles. Thus, a series of circles of uniform diameter may be used to illustrate the unit fractions,

FIGURE 4. THE UNIT FRACTIONS, $\frac{1}{2}$ TO $\frac{1}{9}$, INCLUSIVE



$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$ and to show that $\frac{1}{2}$ is larger than $\frac{1}{3}$, that $\frac{1}{3}$ is larger than $\frac{1}{4}$, etc. Such experiences lead the pupil to see that *the larger the denominator the smaller the fraction,* if the numerator remains the same.

A single circle divided by four diameters into eighths illustrates comparatively the size of $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, and $\frac{8}{8}$ and shows that $\frac{2}{8}$ is more than $\frac{1}{8}$, that $\frac{3}{8}$ is more than $\frac{2}{8}$, etc. Here the pupil sees that the denominator tells what is being compared while the numerator tells how many are being compared. The comparison of fractions having like denominators may also be accomplished by expressing the numerators in numeral form and the denominator in verbal form. This

impresses forcibly the fact that when two like fractions are compared, the comparison is between the numerators. Experiences such as this lead pupils to see that *the larger the numerator the larger the fraction* if the denominator remains the same.

1 eighth
2 eighths
3 eighths
etc.

There should be many experiences such as these to enable pupils to see the relationship between the size of the numbers used as numerator and denominator and the size of the fractions.

Exercises in the comparison of fractions also help pupils to understand that when the numerator and the denominator are equal, the fraction is always equal to 1. Thus, they learn that $2/2 = 3/3 = 4/4 = 5/5 = 6/6 = 8/8 = 1$ and that two-halves of a thing, or three-thirds of it, or four-fourths of it, etc., are all of it. This is frequently a difficult point if it is not well illustrated objectively. They also learn that when the numerator is less than the denominator, the fraction is less than 1. A little later, when they study improper fractions, they discover that if the numerator is greater than the denominator, the fraction is more than 1.

Changing the denominator of a fraction. While pupils are comparing fractions, they acquire considerable information as to the equivalence of fractions which have different numerators and different denominators. This leads, eventually, to the discovery of rules about reducing fractions to lower terms, or changing them so that their numerators and their denominators are larger, as in changing them to a common denominator form. Some teachers make the serious mistake of having pupils

reduce fractions to lower terms by simply following a procedure dictated by the teacher and used by them without regard to whether or not they understand it. The pupils are told that they can divide the numerator and the denominator by the same number. They do it, and so long as they do not forget the process their answers are right. But in time their memory of the process may become hazy and they may make ludicrous blunders which might well have been avoided had the process been properly rationalized.

Such semi-concrete materials as the circular and the rectangular diagrams which we have been using illustrate nicely the equivalence of certain fractions. Figure 5 shows very clearly that $2/8 = 1/4$, that $4/8 = 2/4$, that $2/4 = 1/2$, that $4/8 = 1/2$, and that $6/8 = 3/4$.

FIGURE 5. THE EQUIVALENCE OF HALVES, FOURTHS, AND EIGHTHS.

<table>
<tr><td colspan="8" align="center">One</td></tr>
<tr><td colspan="4" align="center">$\frac{1}{2}$</td><td colspan="4" align="center">$\frac{1}{2}$</td></tr>
<tr><td colspan="2" align="center">$\frac{1}{4}$</td><td colspan="2" align="center">$\frac{1}{4}$</td><td colspan="2" align="center">$\frac{1}{4}$</td><td colspan="2" align="center">$\frac{1}{4}$</td></tr>
<tr><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td></tr>
</table>

Figure 6 illustrates in a similar way the equivalence of halves and sixths and of thirds and sixths. From the two diagrams here, the pupil learns that $3/6 = 1/2$, that $2/6 = 1/3$, and that $4/6 = 2/3$.

Many such diagrams can be drawn but not many are needed to enable the pupils to discover the more important equivalences among the more common frac-

tions. When they have discovered from their experiences with such figures the important principle that *the numerator and the denominator of a fraction can be divided by the same number without changing the value*

FIGURE 6. THE EQUIVALENCE OF HALVES, THIRDS, AND SIXTHS

One					
$\frac{1}{2}$			$\frac{1}{2}$		
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

One					
$\frac{1}{3}$		$\frac{1}{3}$		$\frac{1}{3}$	
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

of the fraction, the figures will have served their purpose. It will be well for the teacher to prepare figures showing the equivalence of twelfths and fractions having smaller denominators. There may be a diagram for halves, fourths, and twelfths; one for halves, sixths, and twelfths, and one for thirds, sixths, and twelfths. These will enable the pupils to discover that $2/12 = 1/6$, that $3/12 = 1/4$, that $4/12 = 2/6$, that $4/12 = 1/3$, that $6/12 = 3/6$, that $6/12 = 2/4$, that $6/12 = 1/2$, that $8/12 = 4/6$, that $8/12 = 2/3$, that $9/12 = 3/4$, and that $10/12 = 5/6$.

In preparing exercises for practice in reducing fractions to lowest terms, particular attention should be given to those fractions which are obtained as the result of adding or subtracting fractions. Fourths, sixths, eighths, tenths, twelfths, sixteenths, twentieths, and twenty-fourths will occur fairly frequently. Others, such as ninths, fourteenths, fifteenths, eighteenths, twenty-fifths, thirtieths, thirty-seconds, forty-eighths, and sixty-

fourths, will occur often enough to justify us in providing some practice on them. Such fractions as twenty-firsts, twenty-seconds, twenty-sixths, twenty-sevenths, twenty-eighths, thirty-sixths, forty-seconds, forty-fifths, etc., will occur so infrequently that it seems best to make them the object of special attention when they do occur, if ever. These groups are more or less arbitrary, to be sure. Other groupings could be made and could be defended. The purpose is to give practice on those fractions which are reasonable sure to occur and to include such a variety of others which may occur that the pupils may see the general application of the principle which we are trying to teach and that they may be able to handle any others which may arise in their experiences.

The first set of practice examples is made up of those mentioned in the first group in the preceding paragraph —fourths, sixths, eighths, tenths, twelfths, sixteenths, twentieths, and twenty-fourths. In this set, there are 52 proper fractions which may be reduced to lower terms. Each of these 52 fractions occurs once in Set 1. These examples would be accompanied by instructions in some such form as the following: Reduce each of these fractions to lowest terms. Remember that sometimes you may divide numerator and denominator by two or more numbers before the fraction is reduced to lowest terms.

Practice Examples in Fractions. Set 1
Reduction for Group One

$4/10 =$	$5/20 =$	$14/24 =$	$6/16 =$	$21/24 =$
$4/6 =$	$8/12 =$	$6/8 =$	$8/24 =$	$12/20 =$
$2/24 =$	$10/20 =$	$12/16 =$	$15/24 =$	$2/4 =$

$\frac{4}{12} =$	$\frac{16}{24} =$	$\frac{8}{20} =$	$\frac{5}{10} =$	$\frac{3}{12} =$
$\frac{3}{6} =$	$\frac{9}{24} =$	$\frac{14}{20} =$	$\frac{2}{8} =$	$\frac{3}{24} =$
$\frac{8}{16} =$	$\frac{6}{20} =$	$\frac{10}{12} =$	$\frac{22}{24} =$	$\frac{8}{10} =$
$\frac{10}{24} =$	$\frac{4}{20} =$	$\frac{15}{20} =$	$\frac{4}{8} =$	$\frac{20}{24} =$
$\frac{2}{12} =$	$\frac{12}{24} =$	$\frac{4}{16} =$	$\frac{18}{20} =$	$\frac{2}{10} =$
$\frac{4}{24} =$	$\frac{16}{20} =$	$\frac{10}{16} =$	$\frac{6}{12} =$	$\frac{2}{6} =$
$\frac{14}{16} =$	$\frac{6}{24} =$	$\frac{6}{10} =$	$\frac{9}{12} =$	$\frac{2}{20} =$
$\frac{18}{24} =$	$\frac{2}{16} =$			

Set 2 contains examples from the second of the groups indicated above—ninths, fourteenths, fifteenths, eighteenths, twenty-fifths, thirtieths, thirty-seconds, forty-eighths, and sixty-fourths. Since several of these denominators are rather large numbers, there are in this group many more fractions that can be reduced than there were in the first group. The total number of proper fractions in this group which can be reduced to lower terms is 127. Of these, 21 are thirtieths, 31 are forty-eighths, and 31 are sixty-fourths. It does not seem necessary to include the entire 127 in the lists which we assign for practice, for if a pupil is proficient with those in Set 1 and can satisfactorily reduce a miscellaneous sample of those with a given denominator in the second group, he should be able to do others in this group without practice. That is, we rely upon transfer. If, for example, a pupil can reduce $\frac{36}{64}$ to lowest terms, he should not have trouble with $\frac{28}{64}$ if he encounters it. Set 2 contains 50 of the 127 fractions. These are selected from the various denominator groups so as to give the pupil practice on a wide variety of examples.

The examples of Set 2 may be given with these instructions. In reducing these fractions to lowest terms, remember that sometimes you can divide numerator

and denominator by two or more numbers. To save time, divide by the largest number that you can see that will divide both numerator and denominator evenly.

PRACTICE EXAMPLES IN FRACTIONS. SET 2
Reduction for Group Two

$\frac{6}{15} =$	$\frac{14}{64} =$	$\frac{4}{14} =$	$\frac{5}{25} =$	$\frac{28}{32} =$
$\frac{15}{48} =$	$\frac{5}{30} =$	$\frac{12}{18} =$	$\frac{48}{64} =$	$\frac{10}{14} =$
$\frac{30}{48} =$	$\frac{10}{25} =$	$\frac{21}{30} =$	$\frac{16}{32} =$	$\frac{6}{18} =$
$\frac{12}{15} =$	$\frac{3}{9} =$	$\frac{24}{64} =$	$\frac{12}{32} =$	$\frac{24}{48} =$
$\frac{12}{30} =$	$\frac{4}{18} =$	$\frac{9}{15} =$	$\frac{12}{14} =$	$\frac{20}{64} =$
$\frac{9}{48} =$	$\frac{20}{32} =$	$\frac{20}{25} =$	$\frac{26}{30} =$	$\frac{5}{15} =$
$\frac{8}{64} =$	$\frac{16}{48} =$	$\frac{10}{18} =$	$\frac{30}{32} =$	$\frac{6}{14} =$
$\frac{6}{48} =$	$\frac{15}{18} =$	$\frac{10}{15} =$	$\frac{56}{64} =$	$\frac{10}{30} =$
$\frac{24}{32} =$	$\frac{6}{9} =$	$\frac{36}{64} =$	$\frac{15}{25} =$	$\frac{6}{32} =$
$\frac{3}{18} =$	$\frac{42}{48} =$	$\frac{2}{14} =$	$\frac{24}{30} =$	$\frac{3}{15} =$

In the instructions for Set 2, it is suggested that the pupil divide by the largest number which he can see will be contained in both numerator and denominator without remainder. As a matter of convenience and economy of time, the teacher may quite properly urge the pupil to look for such numbers as 4, 6, and 8 as divisors but the point is not important enough to justify the teacher in insisting that only these larger numbers be used when possible. We want the pupil to be able to reduce his fractions to lowest terms and to be sure of the accuracy of his results. If he reduces $\frac{24}{64}$ to $\frac{3}{8}$ by dividing by 2 three times instead of by 8 once, no serious loss is suffered.

The largest number which will divide two or more others without remainder is called the greatest common

divisor of the numbers. To reduce a fraction to lowest terms in one operation, then, we divide the numerator and the denominator by their greatest common divisor. There was a day when classes in arithmetic spent considerable time on a formal method for finding the greatest common divisor but that day has gone.

Some teachers have pupils learn and apply a long list of rules for divisibility. One writer recommends rules for divisibility by 2, 3, 4, 5, 6, 9, 10, 12, and 15. This much detail would seem to be more of a burden than a help in the intermediate grades. Pupils should soon learn that a number is divisible by 2 if it is an even number and that even numbers are those ending in 2, 4, 6, 8, and 0. Sometimes pupils try 2 on a fraction such as $^{18}/_{45}$ when they should see at once that 45 is an odd number and, therefore, not divisible by 2. Pupils also may be expected to learn that a number is divisible by 5 if it ends in 5 or 0 and that it is divisible by 10 if it ends in 0. Beyond these three simple rules, there would seem to be little advantage in learning rules for divisibility. Some recommend the rule that a number is divisible by 3 if the sum of its digits is divisible by 3 but in the fractions that occur in the affairs of most persons, there are very few with which this rule could be used to advantage. Numerators and denominators so large that one can not see at once that 3 is a factor of each almost never occur and if they do, one can try 3 on both the numerator and the denominator about as easily as he can add the digits and divide the sums by 3. The rule that a number is divisible by 4 if the number formed by the last two digits is divisible by 4 is also of little value for it is a rare fraction indeed that has more than two

digits in its numerator or in its denominator. As was stated in a preceding paragraph, when one does encounter a fraction with a large denominator, ordinarily he should convert it into a decimal.

The fractions supplied to pupils for practice in reducing to lowest terms should include some which are not reducible. In this case pupils should be warned that there may be such fractions in the list. It should not take a pupil very long to discover that $^{12}/_{25}$ can not be reduced. He will discard 2 as a possible divisor at once and try 3 and discard it readily enough if he knows his division facts. He soon sees that 4 will not do for the denominator, that 5 will not divide the numerator, and that 6 is also impossible for the denominator. Later, he may learn that 4 and 6 will not do if their factors, 2 and 3, could not be used but for the present we can afford to let him try 4 and 6. It is well to point out that in trying various divisors, we never need to go beyond half of the numerator, if the fraction is a proper fraction.

It will be noted that some of the fractions given in the two sets of practice examples will not be obtained as the sum or the difference of two fractions and that it is improbable, therefore, that they will occur at all. Some of these may be obtained in multiplication or division, however, if the pupils do not cancel. Others merely provide the pupils with a wider variety of examples for practice.

While learning to reduce fractions to lower terms, the pupils should also discover that they can perform the opposite operation. That is, they can multiply both numerator and denominator by the same number with-

out changing the fraction's value. This will be evident from the operation of reducing to lower terms. Thus, since $2/4 = 1/2$, $1/2$ also equals $2/4$. Likewise, $1/2$ is the same as $3/6$, as $4/8$, as $5/10$, etc. It does not seem to be worth while, at this stage of their progress, for the pupils to do much work of this kind with fractions but the ease with which the operation may be performed should be seen while reduction to lower terms is being learned. Later, this subject will be given more specific attention in connection with the learning of addition and subtraction of fractions.

QUESTIONS AND REVIEW EXERCISES

1. How early can a pupil be expected to have some understanding of fractions?

2. In general, did Polkinghorne's investigation reveal that the pupils tested knew more or less about fractions than you would have expected?

3. Since children seem to acquire considerable information about fractions before a study of fractions is made in school, why should teachers of the primary grades be concerned about the teaching of fractions?

4. Why should the teacher of an intermediate grade take an inventory of what the pupils already know and understand before beginning a systematic program of instruction in fractions?

5. Have you known pupils who had acquired erroneous ideas about fractions? What, in your opinion is the cause of these erroneous ideas? What should the teacher do to correct them?

6. Why must the teacher use the interview technique in taking an inventory?

7. What is the place of concrete and semi-concrete materials in developing an understanding of fractions? Upon

what theory of instruction is it recommended that the teaching of fractions should be based?

8. Do you believe that the fraction forms, ½, ⅛, ¼, etc., should be learned in the primary grades? Can pupils acquire an understanding of these fractions without learning the numerical forms of representation?

9. What is a prime number? A composite number? Should these terms be learned by pupils in the intermediate grades? Why should they be known by teachers?

10. Which is more meaningful to young children, one-half of a single object or one-half of a group of objects?

11. How may unit fractions be used as an alternative form for expressing the division facts? What is the measurement idea of division? The partition idea?

12. When should the remainder in a division example be expressed as a fractional part of the quotient?

13. How can the facts which pupils learn about measures be used to increase their understanding of fractions?

14. When does the pupil come to a full and complete understanding of what a fraction is? Must a teacher be acquainted with all fraction forms to teach fractions?

15. Why do teachers of today pay less attention to definitions at the beginning of the pupil's study of fractions than did the teachers of years ago?

16. Do definitions serve a useful purpose? If so, to whom and when?

17. How do you account for the English practice of calling common fractions "vulgar fractions?" What does "vulgar" mean when used in this manner?

18. Would you confine school instruction to fractions in common use in the neighborhood? Why?

19. If pupils are competent in the operations with fractions, what objection can there be to the use of examples and problems such as those quoted from Smith?

20. How should a pupil be expected to know that $\frac{1}{6}$ is less than $\frac{1}{2}$? How would you teach pupils the comparative sizes of unit fractions?

21. Which is larger, $\frac{149}{151}$ or $\frac{249}{251}$? How can you tell?

22. How would you help pupils to discover that if the numerator and the denominator of a fraction are divided by the same number the value of the fraction is unchanged? When you were in the grades, did you discover this principle or was it told to you by a teacher?

23. It is sometimes said that the equivalence of certain pairs of fractions, such as $\frac{2}{4}$ and $\frac{1}{2}$, $\frac{4}{8}$ and $\frac{1}{2}$, $\frac{2}{8}$ and $\frac{1}{4}$, etc., should be so well known by the pupils that they would recognize instantly that the one member of a pair is equivalent to the other without reducing. Do you agree?

24. In teaching the reduction of fractions, should the teacher expect any transfer? Why?

25. Do you agree that in reducing fractions to lowest terms the use of the greatest common divisor is a matter of convenience and not a matter of vital importance?

26. What rules for divisibility would you expect pupils to know? How would you teach these rules?

27. How should a pupil discover whether or not a fraction is reducible?

28. Is it worth while for pupils to learn to change fractions to higher terms as well as to reduce them to lower terms?

29. Have you known pupils to make absurd mistakes with fractions because they did not understand fractions? Give examples.

30. When would you expect pupils to begin using the terms, *numerator, denominator, proper, improper,* and *mixed number?*

CHAPTER TEST

For each of these statements, select the best answer. A key will be found on page 533.

1. The pupil's first acquaintance with fractions is usually gained (1) in the primary grades (2) in the intermediate grades (3) out of school.

2. Miss Polkinghorne found that what young children know about fractions was (1) less than (2) equal to (3) more than what would normally have been expected.

3. The youngest children who have some understanding of fractions are of (1) kindergarten age (2) first-grade age (3) second-grade age.

4. The best inventory is accomplished through the use of (1) the interview technique (2) group tests (3) both interviews and group tests.

5. Pupils should gain an understanding of fractions (1) before adding them (2) while adding them (3) after adding them.

6. In the fifth grade, the teacher should begin (1) by reviewing the work of the preceding grade (2) where the pupils are (3) by teaching the new work assigned for the year.

7. Instruction in fractions should be based upon (1) the drill theory (2) the meaning theory (3) the theory of incidental learning.

8. A rectangle can be divided conveniently into smaller rectangles to show the meaning of one-sixth in (1) two ways (2) three ways (3) four ways.

9. The number of prime numbers less than 10 is (1) more than (2) equal to (3) less than the number of composite numbers less than 10.

10. A unit fraction is one (1) whose value is equal to 1 (2) whose numerator is 1 (3) whose denominator is 1.

11. One-half of one is understood by young children

(1) more easily (2) equally easily (3) less easily than one-half of a group.

12. When finding one-fourth of 24, the pupil is using (1) the measurement idea (2) the partition idea (3) the ratio idea of division.

13. The remainder in division should be expressed as a fractional part of the quotient (1) in all of the examples (2) in part of the examples (3) in none of the examples.

14. If definitions are learned, they should be learned (1) before the pupil has experience with fractions (2) while he is having his early experiences with fractions (3) after he has had much experience with fractions.

15. Compared with the ability to identify the various kinds of fractions, definitions of these kinds are (1) less important (2) equally important (3) more important.

16. Historically, common fractions appeared (1) before (2) simultaneously with (3) after decimal fractions.

17. Wilson's investigation revealed that a list of ten commonly occurring fractions made up about (1) 50 per cent (2) 75 per cent (3) 95 per cent of all the uses of fractions which he found.

18. The first work in the comparison of fractions should be developed by (1) changing the fractions to common denominator form (2) using diagrams (3) changing the fractions to decimal form.

19. The expression $4/4$ is (1) a proper fraction (2) an improper fraction (3) an integer.

20. Pupils should learn to reduce fractions to lower terms by (1) discovering the procedure from objective representations (2) following the teacher's directions (3) getting the rule from a textbook.

21. It was recommended that all pupils should learn how to tell when a number is divisible by (1) 2 (2) 3 (3) 4.

22. The subject of highest common factor is emphasized

in the schools of today (1) more than (2) the same as (3) less than was the case a half-century ago.

23. Ordinarily, a fraction with a large denominator (1) should be dropped (2) should be allowed to remain as it is (3) should be converted into a decimal.

24. The value of a fraction is unchanged if (1) the same number is added to both the numerator and the denominator (2) the numerator and the denominator are multiplied by the same number (3) the same number is subtracted from both the numerator and the denominator.

25. The fraction $^{247}/_{249}$ is (1) equal to (2) smaller than (3) larger than the fraction $^{347}/_{349}$.

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8. Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937, 591 pp. The reader is referred to pages 82-92 and 373-389 for a valuable discussion of the teaching of the elements of fractions.

9. Wildeman, Edward. *The Teaching of Fractions.* Chicago: The Plymouth Press, 1923, 145 pp. A rather detailed discussion of the elements of fractions will be found in Chapters I and II.

10. Wilson, Guy Mitchell. *What Arithmetic Shall We Teach?* Boston: Houghton Mifflin Company, 1926, 149 pp. The writer's findings on uses of fractions appear on pages 24-29. Results of a study of use by Carl T. Wise are summarized on pages 55-56. Fractions used in vocational figuring, as reported by a committee headed by Bertha M. Hatch, are reported on pages 110-111.

11. Wilson, Guy M. and Dalrymple, Charles O. "Useful Fractions." *Journal of Educational Research,* XXX: 341-347, January, 1937. Suggests the fractions which should be emphasized in teaching and the operations with fractions which are required in business affairs.

CHAPTER 5

ADDITION AND SUBTRACTION WITH FRACTIONS

In Chapter 4, the importance of developing an understanding of fractions before an effort is made to add, subtract, multiply or divide with them was emphasized. There are many pupils in the fifth, the sixth, and later grades whose progress in learning the operations with fractions has been seriously hindered by the fact that they had not had a sufficient background of meaningful experiences with fractions before these operations were undertaken.

The teacher who takes an inventory of what pupils understand about fractions and can do with them often discovers that some of the pupils are not ready for the work which is scheduled for the year. Some may have only a slight deficiency in their preparation but others may know so little about the subject that none of the scheduled lessons can be meaningful to them. Of course, the progress which they have been able to make has been conditioned by their general intelligence and their special aptitude for arithmetic, but the kind and amount of teaching which they have had is also a very important factor. There are many teachers who seem to be unwilling to accept responsibility for the growth of fraction concepts among pupils in the primary grades. As a consequence, groups of pupils who differ but little in intelligence and aptitude for arithmetic are sometimes found to differ markedly in the progress which they have made by the time they enter the grade where fractions are very much the order of the day.

Should the teacher whose inventory reveals these deficiencies be responsible for aiding her pupils in gaining the concepts which they need but have failed to gain? Is it the duty of a teacher to try to make good the shortcomings of her predecessors? There are teachers who are very reluctant to admit that they have any responsibility beyond that prescribed by the course of study for the grade in which they teach. But it is a shortsighted policy, especially in arithmetic, to undertake a program of lessons with pupils who are not ready for them. Such a program will not be successful. Even though time is lost from the regular program in making up deficiencies in the pupils' preparation, time will be gained in the long run for the pupils will make much greater progress in the work which lies ahead with than without an adequate foundation. Indeed, without such a foundation, satisfactory progress is impossible. Instruction can not be built upon the meaning theory if the pupils are not yet prepared for that instruction. The learning which the pupils do acquire, if any, will be largely rote learning.

THE ADDITION OF FRACTIONS

A survey of attainment in fractions. The author has made an extensive investigation of the abilities of pupils in addition, subtraction, multiplication and division of fractions in grades five, six, seven, and eight. Twelve tests in fractions were given to pupils in 21 Ohio cities. The first test was a test in the addition of two simple fractions expressed in the equation form. Such items as $\frac{1}{2} + \frac{1}{4} =$, $\frac{1}{4} + \frac{1}{8} =$, $\frac{1}{2} + \frac{2}{5} =$, and $\frac{2}{3} + \frac{3}{8} =$, were used. Twenty-four such examples were

mimeographed on a sheet of paper and the pupils were given three minutes to solve as many as they could. Table 3 shows, for grades five, six, seven, and eight, the per cent of the pupils who scored zero, the median number right, and the maximum score.

It will be seen that the median number right was 1.3 in the fifth grade and that 47 per cent of these pupils were unable to solve any of the examples in the test. This was not very surprising since most of the pupils had but recently begun the study of addition of fractions. However, the results secured in grades six, seven, and eight are by no means satisfactory. The median number right was 6.7 in the sixth grade, a little better in the seventh grade, and not quite so good in the eighth grade. Later remedial instruction in one of these schools produced very pronounced gains in the median number right. The really distressing part of the picture, however, is found in the per cents who scored zero. About one-fourth of the sixth-grade pupils and more than one-third of those in the eighth grade were unable to solve a single example in the test. At the other extreme, there were a few who completed correctly all of the examples in the test and who could have done more examples.

TABLE 3. THE RESULTS OF A TEST IN THE ADDITION OF FRACTIONS

Grade	Per Cent Who Scored Zero	Median Number Right	Highest Score
5	47	1.3	23
6	24	6.7	24
7	29	7.5	24
8	36	6.2	24

A study of pupils' errors. The papers of one class were analyzed in an effort to determine what errors had been made. The class made, in all, 268 errors. It was possible to determine in 262 of the 268 cases the nature of the error. The results of this analysis are shown in Table 4.

It will be seen that performing the wrong operation was the chief error in this class. Instead of adding, these pupils frequently subtracted, multiplied, or divided. Of these three wrong operations, multiplication was the one most frequently performed. All of the examples of the test were given in the equation form with a + sign between the two fractions and it seemed that some of the pupils tended to confuse the + and the × signs. This belief is strengthened by the fact that on the multiplication test, many of the pupils who performed the wrong operation, added. The majority of the instances of performing the wrong operation occurred in the papers of pupils who followed one wrong operation consistently. Some, however, performed various wrong operations with none more prominent than the others. Others, although adding most of the time, occasionally subtracted, multiplied, or divided.

As an illustration of the extent to which pupils sometimes mix wrong operations in a single test, a specific case will be described. One paper in addition showed clear evidence of both multiplication and division, the two operations sometimes occurring in consecutive examples, as shown by these: $5/6 + 5/12 = 1/2$; $2/3 + 3/5 = 6/15$. This pupil tended to divide when division without remainder was possible, as shown by the former of the two examples. However, he did not actually perform

TABLE 4. ERRORS IN THE ADDITION OF FRACTIONS

Types of Errors	Illustrations	Per Cent of Total
Wrong operation performed	$\frac{2}{3} + \frac{3}{5} = \frac{6}{15}$	45.0
Denominators added	$\frac{1}{2} + \frac{2}{5} = \frac{3}{7}$	30.4
Error in computation	$\frac{3}{4} + \frac{2}{5} = \frac{16}{20} = \frac{4}{5}$	13.6
Failure to convert both addends to common denominator	$\frac{2}{3} + \frac{3}{8} = \frac{19}{24}$	1.1
Failure to record denominator of sum	$\frac{1}{2} + \frac{2}{5} = 9$	2.3
Failure to reduce answers to mixed numbers	$\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$	1.1
Failure to reduce answers to lowest terms	$\frac{1}{6} + \frac{1}{2} = \frac{4}{6}$	2.3
Miscellaneous		1.9
Unknown		2.3
Total		100.0

the operation of division by a fraction; he simply divided numerator by numerator and denominator by denominator. The only right answer on this paper surely was gotten by chance. It occurred in the example, $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Adding numerators for a new numerator and de-

nominators for a new denominator is a very common error. It arises from the fact that the pupils have little understanding of fractions, do not know how to add them, and, naturally, react to the familiar + sign in the only way they know. Pupils who understand the meaning of fractions and who are well taught in the procedure of adding them do not make this error. In this particular class, a brief period of remedial instruction followed the test and then the test was repeated. An analysis of the errors made on the second test showed only one case of adding denominators.

The two types of errors which have been discussed account for more than three-fourths of all the errors made. Next in frequency is errors in computation, making up 13.6 per cent of the total. Of course, errors of this class are the result of inadequate instruction or insufficient practice in the fundamental operations with integers.

The remaining types occurred much less frequently. Usually, they represent errors which were found in the paper of only one pupil. A little individual instruction will readily eliminate them in most cases.

The validity of error studies. The reader may well question the validity of such error studies as that just reported. In the first place, only seven types of errors were recognized, in addition to a few which are classified in Table 4 as "Miscellaneous." Other analyses of errors in the addition of fractions have resulted in many more classes of errors.[1] It should be noted, however, that the test here reported included proper fractions

[1] Brueckner, Leo J. "Analysis of Errors in Fractions." *Elementary School Journal*, XXVIII: 760-770, June, 1928.

only, and that there are only two addends in each example. This reduces considerably the number of types of errors. Furthermore, if a pupil made two errors in one example, the two errors were tabulated in their respective classes and not as a separate class to represent this particular combination of errors.

A second reason for doubting the validity of such error analyses is the fact that the analysis was made from the test papers only. It was pointed out in Chapter 4 that an examination of pupils' test papers reveals only partially their faults and questionable work habits. Olander expresses doubt as to the validity of analyses of errors made from test papers.[2] His data on errors made in the operations with integers indicate that teachers are very often wrong in their conclusions when their diagnoses are made from test papers only. Another investigation by Brownell and Watson revealed the superiority of the personal interview technique over the analysis of test papers as a method of diagnosis.[3] They used a test in the addition of proper fractions.

The author does not recommend that teachers restrict themselves to test papers in making analyses of pupils' errors. Information obtained from the test papers should be checked and supplemented by information obtained from individual conferences with the pupils. However, the author has reason to believe that the analysis offered for this particular class is substantially correct. He has found that much practice in error

[2] Olander, Herbert T. "The Need for Diagnostic Testing." *Elementary School Journal*, XXXIII: 736-745, June, 1933.

[3] Brownell, William A. and Watson, Brantley. "The Comparative Worth of Two Diagnostic Techniques in Arithmetic." *Journal of Educational Research*, XXIX: 664-676, May, 1936.

analysis contributes greatly to the skill of the one making the analyses. Later contacts with this particular class supported the belief that the analysis which had been made was approximately correct.

The reader should not conclude that the errors reported here are typical of grade classes in general. Classes will be found to differ markedly in the kinds of errors commonly made and in the frequency of each. This analysis is offered as an illustration of the kind of thing which is very much worth the time and effort of the teacher and not as an indication of what one should expect to find if the papers of another class are analyzed.

As an illustration of the difficulty which is sometimes experienced in analyzing errors and of the inferences which one will draw, a difficult specific case will be described. A pupil obtained $1/19$ as the sum of $2/3$ and $3/5$. The correct sum, $1\ 4/15$, supplies a clue, for the sum, before reducing, is $19/15$. There was no written work on the paper to indicate how the example was solved. Possibly the pupil added the fractions correctly, doing the work "mentally," and simply set down for his answer the reciprocal of the numerator of the sum. This hypothesis was strengthened by the fact that there were in the same paper two more just like the example already given, $5/6 + 5/12 = 1/15$ and $1/3 + 1/6 = 1/3$. Note that the answer for the first of these examples, before reducing, is $15/12$ and for the second, $3/6$. Furthermore, in six additional examples this pupil gave an answer which, in each case, was the numerator of the sum, as $3/4 + 3/16 = 15$. This pupils' score was 9-0, that is, 9 attempted and 0 right. A little analysis followed by individual instruction cleared up his whole difficulty for,

after all, none of his solutions was far from correct although all were wrong.

One boy in the class whose errors have been described scored 20-0 (that is, 20 attempted and none right). An examination of his paper soon revealed that in each of the 20 examples he had divided. Furthermore, in 19 of the 20 examples, the division was correctly done. The same boy scored 12-12 on the subtraction test, 18-16 on the multiplication test, and 14-8 on the division test. In five of the six examples which he missed on the division test, he performed the operation of multiplication. Obviously, there is only a slight difference between the performance of this boy and a performance which rightfully would be called "very good."

A girl in the same class is another interesting case. She scored 24-0. It will be remembered that there were only 24 examples in the test. She multiplied in each case. A score of zero might easily have been one of the highest scores secured.

The data and the illustrative cases which have been presented suggest the interesting and greatly varying performances of pupils who are trying to add fractions. They indicate also that prior to undertaking to add fractions, the pupils should be competent in the fundamental operations with integers and that they should have had much concrete experience with fractions as a means of developing an adequate understanding of them. They show, also, the importance of making very clear to the pupils which operation is to be performed. If the equation form is to be used, the signs must be well known. As is pointed out in a later paragraph, how-

ever, it seems better to make much less use of the equation form of statement.

An elaborate analysis of errors made by 200 pupils in grades 5A, 6B, and 6A at Minneapolis has been reported by Brueckner.[4] He found 6,202 errors in addition of fractions. One hundred sixty-seven of these were instances of omitting the example; these could hardly be called errors. In 1,268 others, he was unable to determine from an examination of the test paper what the difficulty was. The other errors may be grouped as follows:

1. Lack of understanding of the process	1,254 errors or 20.2%
2. Error in reducing to lowest terms	1,088 errors or 17.5%
3. Error in changing to mixed number	1,061 errors or 17.1%
4. Error in computation	855 errors or 13.8%
5. Miscellaneous	509 errors or 8.2%

This distribution probably represents more accurately what one would expect to find in a typical class than does the distribution previously reported since this distribution is based upon far more cases. However, teachers may find distributions which differ markedly from either of those reported due to the special conditions under which the pupils in a particular class have worked.

Types of examples. All examples in the addition of fractions may be classified conveniently into two groups:

[4] Brueckner, Leo J., *loc. cit.*

1. Fractions having the same number as denominator.
2. Fractions having different numbers as denominators.

A great many subordinate classes may be recognized. If the fractions are similar fractions, for instance, we may have a sum which is a proper fraction but not reducible, as $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$; a sum which is a proper fraction and reducible, as $\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$; a sum which is an improper fraction but reducible to unity, as $\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$; a sum which is an improper fraction and reducible to a mixed number but not reducible to lower terms, as $\frac{2}{3} + \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3}$; and a sum which is an improper fraction and reducible to a mixed number and to lower terms, as $\frac{3}{4} + \frac{3}{4} = \frac{6}{4} = 1\frac{2}{4} = 1\frac{1}{2}$ (or $\frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$). Each of these may appear as fractions alone or as mixed numbers. Each of these five classes is represented by a mixed number below, but with different fractions.

(1)	(2)	(3)	(4)	(5)
$3\frac{1}{5}$	$4\frac{1}{8}$	$1\frac{1}{4}$	$3\frac{3}{5}$	$2\frac{5}{6}$
$2\frac{2}{5}$	$2\frac{3}{8}$	$5\frac{3}{4}$	$4\frac{4}{5}$	$3\frac{5}{6}$
$5\frac{3}{5}$	$6\frac{4}{8}=6\frac{1}{2}$	$6\frac{4}{4}=7$	$7\frac{7}{5}=8\frac{2}{5}$	$5\frac{10}{6}=6\frac{4}{6}=6\frac{2}{3}$

Further variations may be made by having three or more addends in each example.

Many more classes of examples can be formed from fractions having unlike denominators. In the following discussion, however, all examples containing unlike fractions are classified roughly, into three groups:

(a) Those in which the least common denominator is the largest denominator given.

(b) Those in which the least common denominator is the product of the given denominators, the denominators having no factors in common.

(c) Those in which the least common denominator is a number smaller than the product of the given denominators, the denominators having one or more common factors.

The addition of similar fractions. Pupils who are ready for the addition of similar fractions should find it no more difficult to add such fractions than to add apples, books, marbles, or pennies. They will see that adding fourths, for instance, is like adding apples, that the *fourths* are the things which are being added and that the numerators tell us how many there are.

We have already recognized five types of examples in the addition of two similar fractions. Since it is presumed that the pupils have already learned to reduce fractions to lowest terms, we can combine the first two of these types for attention in the first lessons on the addition of similar fractions and, a little later, give attention to the remaining three types.

Let us begin with a problem: "If we are in school 3¼ hours in the forenoon and 2¼ hours in the afternoon, how many hours are we in school in a day?"

TEACHER: "How shall we find out?"

PUPIL: "Add 3¼ and 2¼."

TEACHER: "Let us write the numbers on the blackboard as we always do when adding. How many are one-fourth and one-fourth?"

$$\begin{array}{r} 3\tfrac{1}{4} \\ 2\tfrac{1}{4} \\ \hline 5\tfrac{2}{4} \end{array} = 5\tfrac{1}{2}$$

PUPIL: "Two-fourths?"

TEACHER: (Writing ²⁄₄) "And how many are 3 and 2?"

PUPIL: "Five."

TEACHER: "So we are in school how many hours each day?"

PUPIL: "Five and two-fourths."

TEACHER: "But two-fourths can be reduced to what fraction?"

PUPIL: "One-half."

TEACHER: "Then, we can write 5½ as the number of hours we are in school in a day."

Very soon the reduction of 2/4 to ½ will be done mentally and ½ written at once. It seems better to write 5 2/4 at first and to reduce in order to make the first solutions as simple as possible.

Of course, if there is any difficulty, it will occur in adding the fractions, ¼ and ¼. With an adequate background of meaningful experience with fractions, this should not be difficult. If the background of some of the pupils is not adequate, change the problem to one having to do with pies on the restaurant shelf instead of hours, draw diagrams to represent 3¼ pies and 2¼ pies and let the pupils see from the drawings that there are 5½ pies in all.

To help the pupils to understand the addition of ¼ and ¼, it is desirable, sometimes, to write the fractions out by themselves and to modify their form, as follows:

¼	1 fourth
¼	1 fourth
2/4	2 fourths

The pupils add 1 fourth and 1 fourth at the right first and then write 2/4, as the equivalent of 2 fourths, at the left.

Several illustrative examples involving various fractions should be worked out. Then the pupils should be given examples for practice. This early practice should be supervised closely by the teacher lest wrong starts be made and wrong habits developed.

The examples in Set 3 illustrate how such examples may be prepared. In no one of these examples is the sum of the fractions equal to or greater than one. Halves are omitted, therefore, and only one example is given for thirds. The fractions used in the set are thirds, fourths, fifths, sixths, eighths, and twelfths. This set of examples by no means exhausts desirable practice material. Other combinations of fractions may be included and some, such as $\frac{3}{8}$ and $\frac{1}{8}$ may be included again but in different order and with different integers.

PRACTICE EXAMPLES IN FRACTIONS. SET 3
Addition of Similar Fractions

$6\frac{1}{4}$ $\underline{4\frac{1}{4}}$	$3\frac{1}{3}$ $\underline{5\frac{1}{3}}$	$2\frac{1}{8}$ $\underline{7\frac{1}{8}}$	$8\frac{3}{8}$ $\underline{1\frac{1}{8}}$	$9\frac{3}{8}$ $\underline{4\frac{3}{8}}$
$7\frac{1}{5}$ $\underline{3\frac{1}{5}}$	$4\frac{1}{6}$ $\underline{8\frac{1}{6}}$	$6\frac{5}{8}$ $\underline{3\frac{1}{8}}$	$2\frac{1}{12}$ $\underline{5\frac{7}{12}}$	$4\frac{5}{12}$ $\underline{7\frac{1}{12}}$
$6\frac{1}{8}$ $3\frac{1}{8}$ $\underline{5\frac{1}{8}}$	$5\frac{1}{6}$ $7\frac{1}{6}$ $\underline{2\frac{1}{6}}$	$4\frac{1}{8}$ $3\frac{3}{8}$ $\underline{4\frac{3}{8}}$	$2\frac{1}{5}$ $6\frac{1}{5}$ $\underline{4\frac{1}{5}}$	$7\frac{1}{8}$ $4\frac{3}{8}$ $\underline{8\frac{1}{8}}$
$5\frac{1}{4}$ $2\frac{1}{4}$ $\underline{7\frac{1}{4}}$	$4\frac{5}{12}$ $3\frac{1}{12}$ $\underline{6\frac{1}{12}}$	$5\frac{5}{12}$ $3\frac{1}{12}$ $\underline{1\frac{5}{12}}$	$7\frac{1}{8}$ $1\frac{5}{8}$ $\underline{2\frac{1}{8}}$	$6\frac{1}{12}$ $1\frac{7}{12}$ $\underline{5\frac{1}{12}}$

The teacher should also give attention to examples in which some of the addends are integers, such as the following:

	$6\frac{1}{3}$	5	$2\frac{1}{8}$	7
5	3	$1\frac{3}{4}$	$5\frac{3}{8}$	$3\frac{1}{6}$
$\underline{3\frac{1}{2}}$	$\underline{5\frac{1}{3}}$	$\underline{6\phantom{\frac{1}{2}}}$	$\underline{8\phantom{\frac{1}{2}}}$	$\underline{2\frac{1}{6}}$

Such examples arise out of real problem situations and must frequently be solved. If neglected, the pupils sometimes have difficulty with them.

It will be seen that we have provided for the introduction of addition of fractions by using examples containing mixed numbers rather than isolated fractions. There are two reasons for this. In the first place, most of our adding of fractions in real life experiences occurs with mixed numbers. Real meaningful problems leading to examples of this kind are easily secured, while it is sometimes very difficult to get interesting practical cases of adding fractions alone. Secondly, it seems to be easier for children to understand the addition of fractions in examples containing mixed numbers. These examples look like examples to which they have long been accustomed except that the fractions have been appended. If fractions alone are added they must be written like this, $\frac{1}{4} + \frac{1}{4} =$, or like this, $\begin{array}{l}\frac{1}{4}\\ \underline{\frac{1}{4}}\end{array}$. The former rarely occurs except in schoolrooms and the latter is likely to be confusing at first appearance.

It would be a mistake, however, to confine our practice work in the addition of fractions to either of these two forms exclusively. Both forms are used, although the column form is used more generally than is the equation form. It has already been suggested that pupils sometimes become confused over the meaning of the

signs. These signs should be learned. Furthermore, standardized tests in arithmetic frequently contain examples in the equation form. If we would have our pupils well trained in the operations with fractions as they occur in tests, in books, and in life situations, then, we should give practice on the two forms of expression which have been given.

While learning to add similar fractions, pupils will practise on both mixed numbers and examples whose addends are fractions only. In the latter case, they will use both the equation and the column forms of statement, as has been indicated. When adding fractions alone, pupils seem to be in greatest danger of forgetting that they are adding fractions and, sometimes, even after good teaching, add the two or more denominators for the denominator of the sum, as $1/8 + 3/8 = 4/16$. The teacher should emphasize frequently the fact that adding eighths is like adding anything else, as pencils, dolls, or chairs, and that $1/8$ and $3/8$ (or 1 eighth and 3 eighths) are $4/8$ (or 4 eighths) just as 1 pencil and 3 pencils are 4 *pencils*.

Changing improper fractions to integers or to mixed numbers. In adding fractions, pupils will soon encounter examples in which the sum of the fractions is equal to or greater than one. It is not at all likely that improper fractions will be met in such work as properly precedes the learning of the operations. Improper fractions are little used in business and social affairs; mixed numbers are used instead. Such an expression as, "two and a half gallons" is quite meaningful to most persons but "five-halves gallons" would confuse many. In order

that they may make the right disposition of improper fractions obtained as parts of sums, pupils must learn to change improper fractions to mixed numbers.

Of course, the work should begin with concrete and semi-concrete materials. If apples, sheets of paper, etc., are cut into pieces of the proper size, the pupils will easily see that $3/2 = 1\frac{1}{2}$, that $4/3 = 1\frac{1}{3}$, that $5/3 = 1\frac{2}{3}$, that $5/4 = 1\frac{1}{4}$, that $6/4 = 1\frac{2}{4} = 1\frac{1}{2}$, that $7/4 = 1\frac{3}{4}$, etc. The procedure is similar to that used in Chapter 4 for teaching the pupils to reduce fractions to lower terms. Half-dollars and quarters can also be used to excellent advantage in showing how improper fractions whose denominators are 2 or 4 can be expressed as equivalent mixed numbers. The use of the actual coins makes the illustrations meaningful and realistic.

A frequent question should be, "How many halves (or thirds, or fourths, etc.) make a whole?" Then, "If we have five half-apples, and two half-apples make a whole apple, how many whole apples shall we have? And how many halves left over? Then, five half-apples make how many apples?"

This emphasis on the number of halves, thirds, etc., in a whole leads naturally to division of the numerator of an improper fraction by the denominator to change it to a mixed number. The experiences which the pupils have should lead them to discover this rule or, at least, to understand it. They can at least verify it from their own experiences. The rule should not be supplied by the teacher at the beginning and merely learned by rote and followed blindly.

By this means, the pupils are led to see that a fraction

means division, that $6/2$ means 6 divided by 2, etc. In their later work, this concept of a fraction as an indicated division will be strengthened and further amplified.

Carrying in addition of fractions. When the pupils have learned to convert improper fractions to integers or to mixed numbers, they can practise on many examples in the addition of similar fractions which have not yet been used. All three of the remaining types of examples involving similar fractions (see page 224) can be included in sets of practice exercises. So far, we have been unable to use the fractions, $1/2$, $2/3$, $3/4$, $4/5$, $5/6$, $7/8$, etc., because the addition of other halves, thirds, fourths, etc., would have given sums which are improper fractions. That is, we have been unable to use these fractions in combination with other fractions; we have used them in examples like $\begin{array}{r} 5 \\ \underline{3\tfrac{1}{2}} \end{array}$, $\begin{array}{r} 2\tfrac{3}{4} \\ \underline{4\phantom{\tfrac{3}{4}}} \end{array}$, etc.

Set 4 contains further examples for practice in the addition of similar fractions. Halves, thirds, fourths, fifths, sixths, eighths, and twelfths are included in this set. Like Set 3, this set is not exhaustive, but illustrates the kind of examples which may be used.

PRACTICE EXAMPLES IN FRACTIONS. SET 4
Addition of Similar Fractions

$\begin{array}{r} 9\tfrac{4}{5} \\ \underline{4\tfrac{2}{5}} \end{array}$	$\begin{array}{r} 6\tfrac{3}{8} \\ \underline{2\tfrac{5}{8}} \end{array}$	$\begin{array}{r} 5\tfrac{2}{3} \\ \underline{3\tfrac{2}{3}} \end{array}$	$\begin{array}{r} 6\tfrac{1}{2} \\ \underline{4\tfrac{1}{2}} \end{array}$	$\begin{array}{r} 7\tfrac{3}{4} \\ \underline{8\tfrac{1}{4}} \end{array}$
$\begin{array}{r} 1\tfrac{7}{8} \\ \underline{3\tfrac{1}{8}} \end{array}$	$\begin{array}{r} 6\tfrac{3}{4} \\ \underline{3\tfrac{3}{4}} \end{array}$	$\begin{array}{r} 3\tfrac{11}{12} \\ \underline{4\tfrac{5}{12}} \end{array}$	$\begin{array}{r} 1\tfrac{2}{3} \\ \underline{4\tfrac{1}{3}} \end{array}$	$\begin{array}{r} 4\tfrac{1}{6} \\ \underline{7\tfrac{5}{6}} \end{array}$

$\begin{array}{r} 3\frac{1}{2} \\ 5\frac{1}{2} \\ \underline{6\frac{1}{2}} \end{array}$	$\begin{array}{r} 2\frac{5}{8} \\ 4\frac{1}{8} \\ \underline{3\frac{3}{8}} \end{array}$	$\begin{array}{r} 6\frac{1}{3} \\ 7\frac{2}{3} \\ \underline{3\frac{1}{3}} \end{array}$	$\begin{array}{r} 3\frac{1}{4} \\ 2\frac{3}{4} \\ \underline{4\frac{3}{4}} \end{array}$	$\begin{array}{r} 6\frac{5}{12} \\ 2\frac{7}{12} \\ \underline{5\frac{1}{12}} \end{array}$
$\begin{array}{r} 7\frac{1}{8} \\ 4\frac{7}{8} \\ \underline{6\frac{5}{8}} \end{array}$	$\begin{array}{r} 5\frac{3}{4} \\ 1\frac{1}{4} \\ \underline{6\frac{1}{4}} \end{array}$	$\begin{array}{r} 4\frac{3}{5} \\ 2\frac{1}{5} \\ \underline{5\frac{4}{5}} \end{array}$	$\begin{array}{r} 3\frac{2}{3} \\ 4\frac{1}{3} \\ \underline{5\frac{2}{3}} \end{array}$	$\begin{array}{r} 5\frac{3}{4} \\ 6\frac{3}{4} \\ \underline{1\frac{3}{4}} \end{array}$
$\begin{array}{r} 7\frac{2}{3} \\ 4\frac{2}{3} \\ \underline{8\frac{2}{3}} \end{array}$	$\begin{array}{r} 5\frac{11}{12} \\ 4\frac{7}{12} \\ \underline{3\frac{1}{12}} \end{array}$	$\begin{array}{r} 6\frac{5}{8} \\ 2\frac{7}{8} \\ \underline{7\frac{3}{8}} \end{array}$	$\begin{array}{r} 6\frac{1}{3} \\ 7\frac{1}{3} \\ \underline{4\frac{1}{3}} \end{array}$	$\begin{array}{r} 6\frac{1}{6} \\ 3\frac{1}{6} \\ \underline{2\frac{5}{6}} \end{array}$

An example will be placed on the blackboard and the discussion may proceed somewhat as follows:

TEACHER: "How many are three-fourths and three-fourths?"

PUPIL: "Six-fourths."

TEACHER: "Then, what is the sum?"

PUPIL: "Ten and six-fourths."

$$\begin{array}{r} 6\frac{3}{4} \\ \underline{4\frac{3}{4}} \\ 10\frac{6}{4} \end{array}$$

TEACHER: "How can you reduce six-fourths?"

PUPIL: "Reduce it to three-halves."

TEACHER: "And how can you change three-halves?"

PUPIL: "Change it to one and one-half."

TEACHER: "Let us write the ½ and carry the 1 to the next column. 1 and 6 and 4 are 11. Then what is the sum?"

$$\begin{array}{r} 6\frac{3}{4} \\ \underline{4\frac{3}{4}} \\ 11\frac{1}{2} \end{array}$$

PUPIL: "Eleven and one-half."

Of course, the 6/4 may be changed to a mixed number and the remaining 2/4 reduced to lowest terms, if preferred. In some examples, it is easier to reduce the improper fraction to a mixed number first rather than to reduce it to lowest terms (if possible) first. Pupils may

be permitted to perform these two operations in either order.

In some examples, the pupils easily learn to abbreviate the process. In the example shown, for instance, they learn to combine $\frac{1}{3}$ and $\frac{2}{3}$, making 1, and to hold the 1 in mind while writing the remaining $\frac{1}{3}$ as a part of the sum.

$$\begin{array}{r} 4\frac{1}{3} \\ 7\frac{2}{3} \\ 3\frac{1}{3} \\ \hline \end{array}$$

The pupils should grow steadily in the ability to convert improper fractions to mixed numbers and to reduce fractions to lower terms "in their heads" and to confine their writing to the writing of the sum in final form. Many will do this if encouraged to do so by their teachers.

The addition of unlike fractions. By this time, the pupils should be able to add like fractions as they appear alone and in mixed numbers as well as to add a fraction and a whole number, a fraction and a mixed number, or a whole number and a mixed number. They should be competent also in reducing fractions to lowest terms and in changing improper fractions to mixed numbers, the latter including carrying in an addition example.

Since these difficulties have been mastered, it is not necessary to form many separate classes of examples in the addition of unlike fractions. Reduction to lowest terms and the changing of improper fractions to mixed numbers can be taken care of as a matter of course in examples having unlike denominators; so can the combination of fractions and whole numbers, fractions and mixed numbers, and whole numbers and mixed numbers.

This makes it possible to simplify greatly the treatment of addition of unlike fractions. We can recognize

separately the 40 types of examples given by Brueckner,[5] but it is much easier for a teacher to keep in mind a plan in which there are a few major steps, with subordinate steps under each, than one in which there are many steps. We shall consider, then, the three groups of examples in the addition of unlike fractions which have been given on pages 224-225.

The least common denominator given. The background of concrete experience with fractions and the liberal use of such semi-concrete materials as circle and rectangle diagrams which have been stressed, together with ample experience in the addition of like fractions, make the approach to the addition of unlike fractions in which the least common denominator is one of the denominators given, relatively direct and simple. For example, children learn to add halves and fourths quickly and easily if they know from many experiences that ½ is equal to ²⁄₄.

The pupils should see that they can not add unlike fractions as they stand but that they must change them so that they will have the same denominator. A problem may be used and the lesson may proceed as follows:

TEACHER: "On Saturdays, Dick helps Mr. Jackson in the grocery. Last Saturday, Dick worked 3¼ hours in the forenoon and 4½ hours in the afternoon. How many hours did he work that day? How shall we find out?"

PUPIL: "Add 3¼ and 4½."

TEACHER: "Let us write the example on the blackboard.

[5] Brueckner, Leo. J. *Diagnostic and Remedial Teaching in Arithmetic.* Philadelphia: The John C. Winston Company, 1930, pp. 177-180.

We can not add $\frac{1}{4}$ and $\frac{1}{2}$ as they are, but we can change $\frac{1}{2}$ to fourths. One-half equals how many fourths?"

$$\begin{array}{r} 3\frac{1}{4} = 3\frac{1}{4} \\ \underline{4\frac{1}{2} = 4\frac{2}{4}} \\ 7\frac{3}{4} \end{array}$$

PUPIL: "Two-fourths."

TEACHER: "Then, $\frac{1}{4}$ and $\frac{2}{4}$ are how much?"

PUPIL: "Three-fourths."

TEACHER: "Then, what is the sum?"

PUPIL: "Seven and three-fourths."

It should not be difficult for the pupils to follow this solution if the equivalence of halves and fourths is already well known. Indeed, they should soon be able to write the sum at once without re-writing the mixed numbers as shown. If necessary, go back to diagrams to show the equivalence of $\frac{1}{2}$ and $\frac{2}{4}$.

Several examples will be used for illustrative purposes. In these, the fractions will be those whose equivalence has already been discovered through the use of diagrams. From these examples, there will emerge gradually the principle that *both the numerator and the denominator can be multiplied by the same number without changing the value of the fraction.* When learned in this way, the principle is meaningful and is not likely to be forgotten.

Set 5 of the practice examples has been prepared to illustrate examples of this class and to supply practice material for the use of the pupils. In this set, the pupil learns to combine halves with fourths, sixth, eighths, and twelfths; fourths with eighths and twelfths; and to make several different combinations of fractions in examples having three addends. No fractions of extremely rare occurrence are included in this set. Of the fractions used, those of most frequent occurrence in prac-

tical affairs, such as halves, fourths, and eighths, a[r] used most often. No combination of fractions occu[rs] twice in the set.

Practice Examples in Fractions. Set 5
Common Denominator Given

$5\frac{1}{2}$ $3\frac{1}{4}$	$2\frac{1}{3}$ $8\frac{1}{6}$	$7\frac{3}{4}$ $4\frac{5}{8}$	$1\frac{2}{3}$ $3\frac{5}{12}$	$2\frac{5}{6}$ $3\frac{1}{2}$
$6\frac{3}{8}$ $10\frac{1}{2}$	$7\frac{1}{12}$ $4\frac{1}{2}$	$4\frac{3}{4}$ $7\frac{1}{2}$	$6\frac{1}{6}$ $3\frac{1}{2}$	$8\frac{2}{3}$ $4\frac{1}{6}$
$8\frac{1}{2}$ $4\frac{5}{8}$	$6\frac{1}{4}$ $2\frac{3}{8}$	$5\frac{7}{8}$ $2\frac{3}{4}$	$7\frac{3}{4}$ $4\frac{5}{12}$	$7\frac{1}{8}$ $4\frac{1}{2}$
$11\frac{1}{3}$ $8\frac{5}{6}$	$6\frac{1}{4}$ $3\frac{1}{8}$	$9\frac{7}{8}$ $3\frac{1}{2}$	$2\frac{2}{3}$ $1\frac{5}{6}$	$4\frac{1}{8}$ $5\frac{3}{4}$
$5\frac{1}{4}$ $3\frac{1}{2}$ $6\frac{3}{8}$	$10\frac{3}{4}$ $5\frac{1}{2}$ $4\frac{7}{12}$	$2\frac{1}{2}$ $4\frac{1}{4}$ $3\frac{1}{2}$	$4\frac{2}{3}$ $10\frac{1}{2}$ $8\frac{1}{6}$	$4\frac{1}{6}$ $6\frac{5}{6}$ $7\frac{1}{2}$
$12\frac{1}{2}$ $6\frac{2}{3}$ $4\frac{11}{12}$	$2\frac{1}{3}$ $5\frac{1}{6}$ $4\frac{2}{3}$	$4\frac{2}{5}$ $6\frac{1}{2}$ $5\frac{3}{10}$	$6\frac{7}{8}$ $3\frac{1}{2}$ $5\frac{1}{2}$	$5\frac{1}{4}$ $2\frac{1}{8}$ $6\frac{3}{4}$
$2\frac{7}{8}$ $6\frac{1}{2}$ $4\frac{3}{4}$	$5\frac{5}{6}$ $3\frac{1}{3}$ $4\frac{1}{2}$	$7\frac{1}{3}$ $2\frac{3}{4}$ $2\frac{5}{12}$	$7\frac{3}{4}$ $14\frac{1}{2}$ $10\frac{1}{4}$	$11\frac{1}{6}$ $4\frac{1}{2}$ $3\frac{1}{6}$
$4\frac{1}{4}$ $2\frac{1}{12}$ $7\frac{5}{6}$	$3\frac{3}{4}$ $2\frac{7}{8}$ $11\frac{1}{4}$	$3\frac{1}{6}$ $5\frac{1}{3}$ $8\frac{1}{6}$	$16\frac{3}{8}$ $2\frac{1}{2}$ $4\frac{5}{8}$	$7\frac{1}{3}$ $5\frac{5}{12}$ $2\frac{5}{6}$

In time, the equivalence of certain fractions of very frequent occurrence will be memorized. If sufficient

practice is provided and this practice is properly distributed, the pupils should come to see instantly the equivalence of $\frac{1}{2}$ and $\frac{2}{4}$, $\frac{1}{2}$ and $\frac{3}{6}$, $\frac{1}{2}$ and $\frac{4}{8}$, $\frac{1}{3}$ and $\frac{2}{6}$, $\frac{2}{3}$ and $\frac{4}{6}$, $\frac{1}{4}$ and $\frac{2}{8}$, $\frac{3}{4}$ and $\frac{6}{8}$, $\frac{1}{2}$ and $\frac{5}{10}$, $\frac{1}{5}$ and $\frac{2}{10}$, and probably others. All of this takes time, of course; the teacher should expect such knowledge and skill with fractions to be the result of slow growth. But after a few examples, the pupils should not hesitate long in converting $\frac{1}{2}$ to fourths, or $\frac{1}{4}$ to eighths. Others will be learned as additional practice is provided.

Denominators without common factors. The least common denominator is the product of the given denominators if these denominators have no common factors. We do not talk to the pupils about factors but we do proceed next to teach them to add fractions which belong to this class. The teacher may state a problem and develop the solution in the following manner.

TEACHER: "On Saturday afternoon, Elizabeth helps her father in his dry goods store. Last Saturday, she sold $4\frac{1}{2}$ yards of muslin to one lady and $2\frac{1}{3}$ yards to another. How many yards did she sell in all? How shall we find out?"

PUPIL: "Add $4\frac{1}{2}$ and $2\frac{1}{3}$."

TEACHER: "We have not yet learned to add $\frac{1}{2}$ and $\frac{1}{3}$. You will remember that one day we changed $\frac{1}{2}$ to sixths. How many sixths did we have?"

PUPIL: "Three-sixths."

TEACHER: "We also changed $\frac{1}{3}$ to sixths. How many sixths did we have?"

$$\begin{array}{r} 4\frac{1}{2}=4\frac{3}{6} \\ 2\frac{1}{3}=2\frac{2}{6} \\ \hline 6\frac{5}{6} \end{array}$$

PUPIL: "Two-sixths."

TEACHER: "Then we can rewrite the example and add as we have added other examples. What is the sum?"

PUPIL: "Six and five-sixths."

If necessary, reproduce the rectangle diagrams which showed the equivalence of halves and sixths and thirds and sixths. Each pupil should see from such concrete representation that halves and thirds can both be changed to sixths. The rule for finding common denominators should be developed from specific examples; it should not be handed out by the teacher at the very beginning of work on examples of this class.

By reference to the diagrams, the pupils recall or discover that $\frac{1}{2} = \frac{3}{6}$, $\frac{1}{3} = \frac{2}{6}$, and $\frac{2}{3} = \frac{4}{6}$. Additional examples in which halves and thirds are to be added can then be given.

Examples in which thirds and fourths are to be added may come next. Diagrams show that $\frac{1}{3} = \frac{4}{12}$, $\frac{2}{3} = \frac{8}{12}$, $\frac{1}{4} = \frac{3}{12}$, and $\frac{3}{4} = \frac{9}{12}$. Several examples can be made from these four fractions. We can combine $\frac{1}{3}$ with $\frac{1}{4}$, $\frac{1}{3}$ with $\frac{3}{4}$, $\frac{2}{3}$ with $\frac{1}{4}$, and $\frac{2}{3}$ with $\frac{3}{4}$.

From these examples, the pupils can see that the common denominator is always the product of the given denominators. This generalization can then be applied to other examples.

Examples for practice and for further illustrations are given in Set 6. No effort has been made to list these examples in a desirable teaching order. For example, all possible combinations of thirds and fourths may well come together. This set merely illustrates in some detail how such examples may be prepared.

Some of the examples in Set 6 have combinations of fractions which will occur rarely if ever in ordinary af-

fairs. These merely provide the pupils with an opportunity to extend their skills so as to cover a wide variety of situations. The following combinations of two fractions will be found in the set:

Halves and thirds	Fourths and fifths
Thirds and fourths	Halves and sevenths
Halves and fifths	Fifths and sixths
Thirds and eighths	Halves and ninths
Thirds and fifths	

PRACTICE EXAMPLES IN FRACTIONS. SET 6
Common Denominator by Multiplication

5 1/2 3 1/3	6 1/4 2 1/3	1 2/3 7 1/2	6 1/3 8 3/4	9 1/2 4 2/5
8 1/3 4 3/5	7 1/4 3 2/3	4 1/7 3 1/2	1 1/5 5 1/2	6 1/2 2 1/9
2 4/5 10 2/3	12 1/4 6 1/5	7 2/3 5 3/4	1 3/4 2 3/5	11 3/4 6 1/3
7 1/2 4 3/7	2 2/5 4 3/8	8 1/6 4 1/5	1 1/4 5 1/3	[illegible] [illegible]
3 5/8 4 3/5	7 3/5 8 1/2	2 3/4 5 1/5	6 2/3 10 1/4	[illegible] [illegible]
12 1/2 8 4/5 7 1/2	3 2/3 2 1/4 7 3/4	7 1/8 2 1/3 6 1/3	7 2/3 6 1/5 4 1/2	[illegible] [illegible] [illegible]
4 1/3 6 3/4 2 1/3	3 1/8 7 5/8 6 1/2	7 3/4 7 1/3 2 2/5	2 2/3 8 1/2 4 2/3	[illegible] [illegible] [illegible]

$2\frac{1}{3}$	$3\frac{1}{4}$	$7\frac{7}{8}$	$6\frac{1}{3}$	$2\frac{1}{2}$
$4\frac{1}{3}$	$5\frac{4}{5}$	$4\frac{2}{3}$	$7\frac{1}{2}$	$4\frac{1}{8}$
$3\frac{2}{5}$	$4\frac{1}{4}$	$3\frac{5}{8}$	$4\frac{1}{2}$	$11\frac{1}{3}$

The set also contains the following combinations of three fractions.

Halves, halves, and thirds
Halves, thirds, and thirds
Thirds, fourths, and fourths
Thirds, thirds, and fourths
Halves, thirds, and fifths
Halves, thirds, and eighths
Thirds, fourths, and fifths
Halves, halves, and fifths
Halves, fifths, and fifths
Thirds, thirds, and eighths
Thirds, eighths, and eighths
Thirds, thirds, and fifths
Fourths, fourths, and fifths

Of course, we might have included others in each of these lists, as fifths and eighths in the first list, or fifths, fifths, and sixths in the second. There is no way to tell just what combinations of fractions should be included in sets of practice examples. If practice is considerably varied and if the subject is taught so as to emphasize the general principles involved, the ability developed should transfer readily to other similar examples which have not been specifically taught.

Several examples can be derived from each of the above groups. The number varies with the particular fractions which are included in a combination. In using the fractions of any one combination, the teacher may

well use all of the possible examples rather than repeat some and omit others. In the combination, thirds—fourths—fifths, for instance, the following arrangements of fractions are possible.

$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} =$	$\frac{2}{3} + \frac{1}{4} + \frac{1}{5} =$
$\frac{1}{3} + \frac{1}{4} + \frac{2}{5} =$	$\frac{2}{3} + \frac{1}{4} + \frac{2}{5} =$
$\frac{1}{3} + \frac{1}{4} + \frac{3}{5} =$	$\frac{2}{3} + \frac{1}{4} + \frac{3}{5} =$
$\frac{1}{3} + \frac{1}{4} + \frac{4}{5} =$	$\frac{2}{3} + \frac{1}{4} + \frac{4}{5} =$
$\frac{1}{3} + \frac{3}{4} + \frac{1}{5} =$	$\frac{2}{3} + \frac{3}{4} + \frac{1}{5} =$
$\frac{1}{3} + \frac{3}{4} + \frac{2}{5} =$	$\frac{2}{3} + \frac{3}{4} + \frac{2}{5} =$
$\frac{1}{3} + \frac{3}{4} + \frac{3}{5} =$	$\frac{2}{3} + \frac{3}{4} + \frac{3}{5} =$
$\frac{1}{3} + \frac{3}{4} + \frac{4}{5} =$	$\frac{2}{3} + \frac{3}{4} + \frac{4}{5} =$

For the combination, thirds—thirds—eighths, we may have the following arrangements.

$\frac{1}{3} + \frac{1}{3} + \frac{1}{8} =$	$\frac{1}{3} + \frac{2}{3} + \frac{5}{8} =$
$\frac{1}{3} + \frac{1}{3} + \frac{3}{8} =$	$\frac{1}{3} + \frac{2}{3} + \frac{7}{8} =$
$\frac{1}{3} + \frac{1}{3} + \frac{5}{8} =$	$\frac{2}{3} + \frac{2}{3} + \frac{1}{8} =$
$\frac{1}{3} + \frac{1}{3} + \frac{7}{8} =$	$\frac{2}{3} + \frac{2}{3} + \frac{3}{8} =$
$\frac{1}{3} + \frac{2}{3} + \frac{1}{8} =$	$\frac{2}{3} + \frac{2}{3} + \frac{5}{8} =$
$\frac{1}{3} + \frac{2}{3} + \frac{3}{8} =$	$\frac{2}{3} + \frac{2}{3} + \frac{7}{8} =$

Of course, each of these arrangements may appear in an order different from that in which it is given.

Some special attention should be given to examples in which a denominator is repeated. The pupils should see that the common denominator is the product of 2 and 3 in the example shown and that we use the denominator 3 only once. They will readily see that this is so if it is pointed out that we *can* add $\frac{1}{3}$ and $\frac{1}{3}$ first and then add $\frac{1}{2}$ to the sum, $\frac{2}{3}$.

$$\begin{array}{r} 4\frac{1}{3} \\ 2\frac{1}{2} \\ \underline{3\frac{1}{3}} \end{array}$$

Denominators with common factors. The product of the denominators of the fractions to be added (or subtracted) is always *a* common denominator but it is the *least* common denominator only when the given denominators have no factors in common. If the given denominators have factors in common, these common factors must be used only once in finding the least common denominator. Thus, the denominators of the fractions, $\frac{3}{4}$ and $\frac{1}{6}$, have 2 as a common factor. The least common denominator is the product of this 2, the other 2 in 4 and the 3 in 6. Then the least common denominator is 12. It will be noted that if 24, the product of the given denominators, is divided by the common factor 2, the least common denominator is obtained.

In the old-fashioned school, a device was employed for removing the factors which were common to two or more of the given denominators and for finding the least common denominator. This device is illustrated by the following example.

$$\begin{array}{r|l} 2 & \frac{2}{3}+\frac{5}{6}+\frac{3}{4}+\frac{1}{2}+\frac{7}{8}=\frac{\quad}{24}+\frac{\quad}{24}+\frac{\quad}{24}+\frac{\quad}{24}+\frac{\quad}{24}=\text{etc.} \\ \hline 2 & 3,\ 3,\ 2,\ 1,\ 4 \\ \hline 3 & 3,\ 3,\ 1,\ 1,\ 2 \\ \hline & 1,\ 1,\ 1,\ 1,\ 2 \end{array}$$

$$\text{L.C.D.}=2\times2\times3\times2=24.$$

The pupil was instructed to divide by any prime factor which was contained without remainder in two or more of the given denominators, to continue this process as long as possible, and to take as the least common denominator the product of these divisors and the final remainders.

Pupils learned this procedure by rote and followed it blindly. It is doubtful whether their teachers could have told *why* it yielded the least common denominator. In doing the assignments of that day, some such procedure was necessary. Sometimes the least common denominator was a large number but much smaller than the product of the denominators of the addends. But the arithmetic of ordinary affairs does not include such examples. This method of finding the least common multiple of a series of numbers or the least common denominator of a series of fractions has been discarded by recent textbooks and by up-to-date teachers although it is still found occasionally in old-fashioned schools taught by old-fashioned teachers.

The pupil understands by this time that a common denominator is a number which is divisible by each of the denominators in the example. For all ordinary examples, the least common denominator is easily found by a trial and error method. Using such an example as that shown, the pupils may at first suggest that the common denominator is 24. The teacher may permit the solution to be completed with this common denominator. Of course, the sum of the two fractions will be reduced from $2\frac{2}{24}$ to $1\frac{1}{12}$. Then, the teacher may suggest that a smaller common denominator may be used and the pupils are asked to think of a number smaller than 24 which will contain both 4 and 6. Most of them will see readily that 12 is such a number.

$$\begin{array}{r} 5\frac{3}{4} \\ \underline{2\frac{1}{6}} \end{array}$$

Two or three relatively easy examples such as this should be used so that the pupils may see that frequently it is possible to find a common denominator

which is smaller than the product of the denominators of the fractions which are to be added. They learn to call such a common denominator the smallest or *least* common denominator. Then, a more difficult example should be presented for solution. This example may have three addends and it should be one for which the finding of the least common denominator is appreciably more difficult than was the case for the first examples of this class. Such an example as the one shown may be used. Most pupils will not readily see the least common denominator here. For such an example, the pupils will learn to try first the largest denominator given and then its successive multiples until one is found which will contain without remainder each of the other denominators.

$$\begin{array}{r} 4\frac{1}{3} \\ 2\frac{3}{8} \\ \underline{3\frac{1}{2}} \end{array}$$

The explanation to the pupils may be given in some such manner as the following. First, try the largest denominator you see, 8. This denominator will contain 2 but not 3. Then try 2 times 8, or 16. This will contain 2 but not 3. Now try 3 times 8, or 24. This contains both 2 and 3, as well as 8, and is the least common denominator.

This method works and works easily for any examples which the pupils are likely to encounter. Even the longer examples in the older textbooks are readily solved by finding the least common denominator in this manner. The pupils will learn to examine the denominators in an example with this question in mind, "Is there any number, other than 1, which will divide two or more of them?" If so, use the method which has just been described. If not, simply find the product of the given denominators.

It should be noted, however, that finding the *least* common denominator is a matter of convenience and not a matter of vital importance. Pupils should be urged, as a matter of economy of time and energy, to use the least common denominator. But if, on a test, a pupil uses some other common denominator, his score should not be discounted very much. The ability to add fractions is the main objective whether the *least* common denominator or any other common denominator is used.

Examples illustrative of the kinds which may be used for practice are given in Set 7. The 50 examples of this set have been carefully prepared so as to provide considerable variety in the fractions to be added and so as to center the pupil's attention largely on those of more frequent occurrence, such as fourths and sixths, sixths and eighths, halves, fourths, and sixths, etc.

Practice Examples in Fractions. Set 7

Common Denominators by Inspection

$6\frac{1}{4}$	$2\frac{3}{4}$	$7\frac{5}{6}$	$2\frac{1}{8}$	$5\frac{1}{10}$
$3\frac{5}{6}$	$4\frac{1}{6}$	$3\frac{3}{8}$	$1\frac{5}{6}$	$1\frac{3}{4}$
$10\frac{5}{8}$	$6\frac{1}{6}$	$2\frac{1}{4}$	$7\frac{1}{6}$	$4\frac{3}{10}$
$7\frac{5}{6}$	$3\frac{1}{4}$	$5\frac{3}{10}$	$1\frac{7}{8}$	$3\frac{5}{6}$
$2\frac{7}{8}$	$1\frac{5}{8}$	$6\frac{1}{8}$	$1\frac{3}{4}$	$12\frac{1}{10}$
$6\frac{3}{10}$	$2\frac{7}{12}$	$6\frac{1}{6}$	$2\frac{7}{10}$	$6\frac{5}{12}$
$9\frac{3}{4}$	$7\frac{3}{8}$	$5\frac{1}{6}$	$3\frac{5}{6}$	$6\frac{7}{8}$
$8\frac{5}{6}$	$10\frac{1}{6}$	$9\frac{7}{10}$	$4\frac{2}{9}$	$8\frac{5}{6}$

$1\frac{5}{8}$ $\underline{2\frac{1}{10}}$	$6\frac{5}{12}$ $\underline{4\frac{3}{8}}$	$2\frac{1}{6}$ $\underline{7\frac{5}{8}}$	$9\frac{3}{10}$ $\underline{10\frac{7}{12}}$	$4\frac{1}{6}$ $\underline{7\frac{5}{9}}$
$4\frac{1}{4}$ $2\frac{1}{6}$ $\underline{3\frac{1}{2}}$	$7\frac{2}{3}$ $6\frac{1}{2}$ $\underline{3\frac{1}{8}}$	$5\frac{1}{2}$ $6\frac{3}{8}$ $\underline{4\frac{1}{8}}$	$7\frac{2}{3}$ $10\frac{1}{4}$ $\underline{5\frac{1}{6}}$	$3\frac{1}{4}$ $2\frac{1}{6}$ $\underline{5\frac{1}{3}}$
$8\frac{1}{4}$ $2\frac{7}{8}$ $\underline{5\frac{1}{3}}$	$1\frac{5}{6}$ $1\frac{3}{4}$ $\underline{6\frac{3}{8}}$	$5\frac{3}{4}$ $4\frac{5}{6}$ $\underline{8\frac{1}{2}}$	$2\frac{1}{2}$ $1\frac{1}{3}$ $\underline{9\frac{1}{8}}$	$12\frac{2}{3}$ $11\frac{3}{4}$ $\underline{7\frac{5}{6}}$
$4\frac{3}{4}$ $5\frac{2}{3}$ $\underline{1\frac{1}{8}}$	$2\frac{1}{2}$ $1\frac{5}{6}$ $\underline{6\frac{3}{4}}$	$3\frac{2}{3}$ $6\frac{1}{2}$ $\underline{4\frac{3}{8}}$	$11\frac{1}{3}$ $4\frac{1}{6}$ $\underline{8\frac{3}{4}}$	$1\frac{3}{4}$ $2\frac{1}{6}$ $\underline{5\frac{1}{8}}$
$6\frac{1}{2}$ $7\frac{1}{4}$ $\underline{4\frac{5}{6}}$	$5\frac{1}{3}$ $7\frac{1}{2}$ $\underline{3\frac{5}{8}}$	$2\frac{5}{6}$ $8\frac{1}{3}$ $\underline{6\frac{1}{4}}$	$10\frac{1}{2}$ $4\frac{7}{8}$ $\underline{7\frac{2}{3}}$	$6\frac{5}{6}$ $2\frac{3}{4}$ $\underline{8\frac{1}{3}}$
$2\frac{1}{6}$ $3\frac{3}{4}$ $\underline{5\frac{2}{3}}$	$7\frac{3}{8}$ $1\frac{2}{3}$ $\underline{6\frac{1}{4}}$	$5\frac{1}{2}$ $12\frac{1}{3}$ $\underline{3\frac{7}{8}}$	$2\frac{1}{4}$ $4\frac{3}{8}$ $\underline{3\frac{5}{6}}$	$6\frac{2}{3}$ $8\frac{5}{6}$ $\underline{1\frac{1}{4}}$

Finally, it may be noted that a few fraction combinations occur so frequently that the pupils will save time by learning them as facts, just as the various combinations of one-digit numbers were learned as facts. There are not many such combinations of fractions, but perhaps the following might be included in a minimum list.

$$\tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6} \qquad \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{3}{8}$$
$$\tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4} \qquad \tfrac{1}{2} + \tfrac{3}{4} = 1\tfrac{1}{4}$$
$$\tfrac{1}{2} + \tfrac{1}{8} = \tfrac{5}{8} \qquad \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{12}$$

Some pupils learn most of these, as the result of many contacts, without their receiving specific attention by the teacher.

THE SUBTRACTION OF FRACTIONS

It was indicated in Chapter 2 that the problems incident to the teaching of subtraction of whole numbers are somewhat fewer and apparently somewhat simpler than those incident to the teaching of addition of whole numbers. The reason for this lies in the fact that in addition there may be many quantities to deal with whereas in subtraction there are but two quantities,—the minuend and the subtrahend. Reasoning by analogy, one might conclude that the teaching of subtraction of fractions and mixed numbers is simpler and possibly easier than the teaching of addition of fractions and mixed numbers. From the teacher's point of view, the process of subtraction may be simpler and easier than the process of addition but, of course, this is no guarantee that it is either simpler or easier for the pupil. It may be easier for the pupil to find the least common denominator in subtraction than in addition but other serious difficulties inhere in subtraction. Prominent among these is the difficulty of borrowing which seems to be more serious than the difficulty of carrying in addition.

Pupils' errors in the subtraction of fractions. A very detailed analysis of errors in the subtraction of fractions has been reported by Brueckner.[6] It will be recalled that he tested 200 pupils in grades VA, VIB, and

[6] Brueckner, Leo J., *op. cit.*, pp. 203-206 and 215-216.

VIA in six elementary schools at Minneapolis. He found more errors in subtraction than in addition, there being 6,202 errors in addition and 7,511 errors in subtraction. His findings for subtraction are summarized in Table 5.

It will be seen that the chief difficulty which these pupils experienced had to do with the operation of borrowing. Mistakes which were due to this difficulty appeared in several forms. For example, in attempting the example shown, pupils would sometimes disregard the fact that they had borrowed and obtain $3\frac{1}{2}$ for an answer. Others would prefix the borrowed 1 to the numerator of $\frac{1}{8}$ and subtract $\frac{5}{8}$ from $1\frac{1}{8}$. Other fairly common errors consisted of simply combining the borrowed number with the numerator, as in $4\frac{1}{3} - 1\frac{2}{3} = 3\frac{2}{3} - 1\frac{2}{3} = 2$; of borrowing when borrowing was not required and leaving an improper fraction in the remainder, as in $4\frac{1}{3} - 1\frac{1}{3} = 3\frac{4}{3} - 1\frac{1}{3} = 2\frac{3}{3}$; and of disregarding the minuend fraction after borrowing, as in $4\frac{1}{3} - 1\frac{1}{3} = 3\frac{3}{3} - 1\frac{1}{3} = 2\frac{2}{3}$. There were others of less frequent occurrence.

$$\begin{array}{r} 5\frac{1}{8} \\ \underline{2\frac{5}{8}} \end{array}$$

Brueckner found that subtraction was often confused with addition. There were many cases of adding fractions and adding them correctly when subtraction was called for. It was also fairly common to find the fractions added and the whole numbers subtracted or the fractions subtracted and the whole numbers added.

Most of the errors which had to do with reducing to lowest terms were simply cases of failure to reduce fractions in answers to lowest terms.

TABLE 5. ERRORS IN SUBTRACTION OF FRACTIONS (AFTER BRUECKNER)

Class of Errors	Number of Errors	Per Cent of Total
Errors in borrowing	1,828	24.3
Confusion of operations	1,521	20.2
Errors in reducing to lowest terms	1,094	14.6
Lack of understanding of process	1,094	14.6
Errors in changing to common denominator form	626	8.3
Errors in computation	614	8.2
Omissions (not really errors)	419	5.6
Performed only a part of the operation	301	4.0
Errors in copying figures	14	0.2
Total	7,511	100.0

Errors which are classified as due to a lack of understanding of the process of subtraction of fractions appeared in many forms. Brueckner lists 16 forms of this class of errors. More than one-half of these occurred in examples in which a mixed number was to be subtracted from an integer, as in $4 - 2\frac{1}{2}$. The pupil simply subtracted the integers and combined the one fraction with the result, obtaining $2\frac{1}{2}$ as his answer. It seems probable that these pupils had had but little practice on examples of this type. Next in frequency of occurrence came the subtraction of the minuend

fraction from the subtrahend fraction, as in $4\frac{1}{3} - 2\frac{2}{3}$. The pupil wrote $2\frac{1}{3}$ as his remainder. The other forms occurred less frequently.

Of the 626 errors which had to do with changing to the common denominator form, 499, or 80 per cent, were cases of subtracting numerators and then using one of the denominators given for the denominator of the result, as in $\frac{7}{8} - \frac{1}{4} = \frac{6}{8}$.

These data indicate very clearly that additional instruction on several fundamental matters is required before these pupils can be expected to avoid many of their errors. The borrowing process is very much in need of further attention. The confusion of operations suggests incomplete learning of each of them. Leaving fractions in answers which could be reduced is largely a matter of carelessness or the result of a lack of emphasis on the part of the teacher. Insufficient attention to some special forms seems to be back of the failure of several pupils to comprehend fully the process of subtraction of fractions.

Obviously, some of these errors are much more serious than others. Errors due to an inadequate understanding of the processes involved in the subtraction of fractions, for example, represent a serious deficiency in the pupils' previous preparation. On the other hand, failure to reduce answers to lowest terms is an error of much less importance. To be sure, it is not good form to leave unreduced such answers as $\frac{2}{4}$, $\frac{6}{8}$, $\frac{3}{6}$, etc., but some teachers make as much ado about this as about much more serious errors. Some teachers discount a pupil's score on a test as heavily for the solution, $\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$, as for the solution, $\frac{2}{3} - \frac{1}{2} = \frac{1}{1} = 1$.

The author in making case studies of pupils' difficulties in the subtraction of fractions has found some interesting examples of inadequate understanding and consequent confusion. One girl scored 12-0 on a test in the subtraction of fractions alone. In the first two examples, she quite evidently subtracted both numerators and denominators, $5/6 - 1/4 = 4/2$ and $5/8 - 1/2 = 4/6$. The third, $7/8 - 9/16 = 7/8$, is a more difficult case. The only clue is that 16 has been written in pencil on the 9. She probably subtracted denominators, then inverted the subtrahend and subtracted numerators, making an error in computation. The next two were scored "Unknown." At this point, the pupil evidently recalled that she had been taught something about cancelation. She did this example, $3/8 - 1/6 = 1/6$ by canceling the 3 into the 6 and then evidently subtracting denominators for the denominator of the answer. She used the one remaining digit of the numerators for the numerator of the answer, failing to understand that the 3 of $3/8$ has been displaced by a 1. Then in this example, $7/12 - 1/3 = 7/1$, the 3 is canceled into 12, 4 is written beneath but this 4 is ignored in writing the answer. Some further recollection about inversion must have pushed to the foreground of consciousness at this time for in the example, $5/6 - 5/12$, this girl availed herself of the very excellent opportunities for canceling but wrote the result $2/1$. One may easily guess at what was done in $2/3 - 3/5 = 2/5$, but the source of the denominator in the answer of $1/3 - 1/6 = 2/3$, where the second fraction was inverted and the digits rewritten, $1/3 - 6/1$, before cancelation was indulged in, is hard to determine. This example, $7/8 - 3/4$, was rewritten, $7/8 - 4/3$, can-

celed, and the answer $\frac{7}{6}$ ingeniously arrived at by multiplying numerators and adding denominators. In $\frac{1}{2} - \frac{1}{4} = \frac{2}{4}$, the 4 has been rewritten on the 1, canceled, but restored to the denominator in writing the answer.

Another girl with the score, 12-0, subtracted denominators for three examples and then performed the operation of division perfectly, rewriting the second fraction, canceling, and multiplying. These two processes were alternated through the remainder of her work.

Types of examples. It is convenient to recognize the same major types of examples in subtraction that we have already recognized in addition. They are:

1. Fractions having the same number as denominator.
2. Fractions having different numbers as denominators.
   (a) Those in which the least common denominator is the largest denominator given.
   (b) Those in which the least common denominator is the product of the given denominators, the denominators having no factors in common.
   (c) Those in which the least common denominator is a number smaller than the product of the given denominators, the denominators having common factors.

Since the addition of fractions and the subtraction of fractions have much in common, we can discuss the subtraction of fractions in briefer space than was required for the discussion of addition of fractions.

The subtraction of similar fractions. Except for

borrowing, subtraction of fractions is easy for those who have learned the corresponding phases of the addition of fractions. But borrowing is really troublesome, as was indicated by the results of Brueckner's study of errors.

It seems best to begin with a few examples in which no borrowing is required in order that the pupils may gain confidence in their ability to solve such examples. These easier examples may consist of fractions alone, or of mixed numbers, or both. The teacher may begin with a problem and develop the subject in the following manner.

TEACHER: "Elizabeth, who helps her father in his dry goods store found that there were just $4\frac{1}{4}$ yards of silk left in a bolt. That afternoon, she sold $1\frac{3}{4}$ yards. How many yards were left? How shall we find out?"

PUPIL: "Subtract $1\frac{3}{4}$ from $4\frac{1}{4}$."

TEACHER: "Let us write the example on the blackboard. Can we subtract $\frac{3}{4}$ from $\frac{1}{4}$?"

PUPIL: "No."

$$\begin{array}{rl} 4\frac{1}{4} & = 3\frac{5}{4} \\ \underline{1\frac{3}{4}} & = \underline{1\frac{3}{4}} \\ & \ 2\frac{2}{4} = 2\frac{1}{2} \end{array}$$

TEACHER: "Then we must borrow just as we always have borrowed in subtraction. We borrow in subtracting fractions just as we borrow in subtracting whole numbers. How many fourths in a whole yard of silk?"

PUPIL: "Four-fourths."

TEACHER: "Four-fourths and this one-fourth make five-fourths. Now we can subtract $\frac{3}{4}$ from $\frac{5}{4}$. How much is left?"

PUPIL: "Two-fourths."

TEACHER: "And since we borrowed one yard, there are 3 left. So how much silk was left in the bolt?"

PUPIL: "$2\frac{2}{4}$ or $2\frac{1}{2}$ yards."

After other illustrative examples, there should be a period of supervised practice on examples of this kind. The pupils should be able to dispense with much of the writing and write their answers, properly reduced, directly.

The practice examples of Set 8 illustrate the kinds of examples which may be used in subtracting similar fractions. There are 50 examples in this set. In the first 25 examples, no borrowing is required; borrowing is required in each of the remaining 25 examples. It will be noted that there are two instances of subtracting an integer from a mixed number and nine instances of subtracting a mixed number from an integer. Examples requiring the subtraction of a fraction from a fraction, a fraction from an integer, and a fraction from a mixed number are easily prepared. They should be included in practice exercises.

Practice Examples in Fractions. Set 8

Subtraction of Similar Fractions

$\begin{array}{r}4\frac{3}{4}\\ 2\frac{1}{4}\\ \hline\end{array}$	$\begin{array}{r}6\frac{1}{4}\\ 3\frac{1}{4}\\ \hline\end{array}$	$\begin{array}{r}7\frac{1}{2}\\ 1\frac{1}{2}\\ \hline\end{array}$	$\begin{array}{r}5\frac{2}{3}\\ 2\frac{1}{3}\\ \hline\end{array}$	$\begin{array}{r}8\frac{2}{3}\\ 3\frac{2}{3}\\ \hline\end{array}$
$\begin{array}{r}7\frac{7}{8}\\ 3\frac{3}{8}\\ \hline\end{array}$	$\begin{array}{r}12\frac{5}{8}\\ 8\frac{1}{8}\\ \hline\end{array}$	$\begin{array}{r}10\frac{3}{5}\\ 5\frac{2}{5}\\ \hline\end{array}$	$\begin{array}{r}12\frac{1}{2}\\ 6\\ \hline\end{array}$	$\begin{array}{r}8\frac{5}{6}\\ 4\frac{1}{6}\\ \hline\end{array}$
$\begin{array}{r}10\frac{3}{8}\\ 4\frac{1}{8}\\ \hline\end{array}$	$\begin{array}{r}7\frac{3}{5}\\ 5\frac{1}{5}\\ \hline\end{array}$	$\begin{array}{r}6\frac{3}{4}\\ 2\frac{3}{4}\\ \hline\end{array}$	$\begin{array}{r}8\frac{5}{8}\\ 1\frac{3}{8}\\ \hline\end{array}$	$\begin{array}{r}11\frac{1}{6}\\ 7\frac{1}{6}\\ \hline\end{array}$
$\begin{array}{r}14\frac{7}{8}\\ 10\frac{5}{8}\\ \hline\end{array}$	$\begin{array}{r}9\frac{4}{5}\\ 1\frac{3}{5}\\ \hline\end{array}$	$\begin{array}{r}6\frac{3}{5}\\ 2\frac{2}{5}\\ \hline\end{array}$	$\begin{array}{r}7\frac{1}{3}\\ 4\frac{1}{8}\\ \hline\end{array}$	$\begin{array}{r}6\frac{3}{7}\\ 4\frac{1}{7}\\ \hline\end{array}$
$\begin{array}{r}7\frac{5}{12}\\ 4\frac{1}{12}\\ \hline\end{array}$	$\begin{array}{r}8\frac{7}{8}\\ 3\frac{1}{8}\\ \hline\end{array}$	$\begin{array}{r}9\frac{4}{5}\\ 7\frac{2}{5}\\ \hline\end{array}$	$\begin{array}{r}10\frac{5}{8}\\ 7\frac{5}{8}\\ \hline\end{array}$	$\begin{array}{r}7\frac{4}{5}\\ 5\\ \hline\end{array}$

$8\frac{1}{4}$ $4\frac{3}{4}$	$2\frac{2}{5}$ $1\frac{4}{5}$	$7\frac{1}{6}$ $3\frac{5}{6}$	$4\frac{3}{8}$ $2\frac{7}{8}$	$16\frac{1}{8}$ $7\frac{3}{8}$
10 $4\frac{1}{2}$	13 $6\frac{2}{3}$	14 $9\frac{1}{4}$	$8\frac{1}{3}$ $3\frac{2}{3}$	$4\frac{5}{8}$ $1\frac{7}{8}$
$9\frac{4}{7}$ $6\frac{5}{7}$	$7\frac{1}{8}$ $3\frac{5}{8}$	$5\frac{3}{5}$ $2\frac{4}{5}$	8 $4\frac{3}{4}$	6 $2\frac{1}{3}$
$7\frac{1}{5}$ $2\frac{2}{5}$	$6\frac{3}{8}$ $4\frac{5}{8}$	9 $4\frac{1}{8}$	$6\frac{1}{5}$ $1\frac{3}{5}$	6 $2\frac{3}{8}$
$9\frac{1}{8}$ $1\frac{7}{8}$	$6\frac{2}{5}$ $3\frac{3}{5}$	12 $4\frac{7}{8}$	$6\frac{1}{5}$ $2\frac{4}{5}$	10 $3\frac{5}{8}$

The subtraction of unlike fractions. When pupils have learned to add unlike fractions and to subtract similar fractions without and with the borrowing difficulty, there is little left for the teacher to do but to provide practice on the various types of unlike fractions in examples in subtraction. Many of the suggestions which have been offered relative to the teaching of addition of fractions apply equally well to the teaching of subtraction of fractions. Pupils should understand now that they must change the fractions so that they have the same denominator and that borrowing is necessary if the fraction part of the minuend is smaller than the fraction part of the subtrahend.

The least common denominator given. Practice Examples in Fractions, Set 9, include unlike fractions in examples in which the larger of the two given denominators is the least common denominator. In the 40 examples of this set, practice is limited largely to halves and fourths, halves and sixths, halves and eighths, thirds

and sixths, and fourths and eighths. Additional examples, requiring the subtraction of fractions from fractions and fractions from mixed numbers, can easily be prepared.

Practice Examples in Fractions. Set 9

Common Denominator Given

$\begin{array}{r}4\frac{3}{4}\\ \underline{3\frac{1}{2}}\end{array}$	$\begin{array}{r}2\frac{1}{2}\\ \underline{1\frac{3}{8}}\end{array}$	$\begin{array}{r}7\frac{5}{8}\\ \underline{4\frac{1}{2}}\end{array}$	$\begin{array}{r}4\frac{2}{3}\\ \underline{1\frac{5}{6}}\end{array}$	$\begin{array}{r}9\frac{3}{4}\\ \underline{2\frac{7}{8}}\end{array}$
$\begin{array}{r}10\frac{1}{3}\\ \underline{4\frac{1}{6}}\end{array}$	$\begin{array}{r}6\frac{3}{4}\\ \underline{1\frac{1}{2}}\end{array}$	$\begin{array}{r}7\frac{1}{6}\\ \underline{4\frac{2}{3}}\end{array}$	$\begin{array}{r}2\frac{3}{8}\\ \underline{1\frac{3}{4}}\end{array}$	$\begin{array}{r}5\frac{1}{2}\\ \underline{2\frac{7}{8}}\end{array}$
$\begin{array}{r}11\frac{7}{8}\\ \underline{4\frac{1}{4}}\end{array}$	$\begin{array}{r}3\frac{1}{3}\\ \underline{1\frac{5}{6}}\end{array}$	$\begin{array}{r}6\frac{1}{8}\\ \underline{3\frac{3}{4}}\end{array}$	$\begin{array}{r}9\frac{3}{4}\\ \underline{4\frac{5}{8}}\end{array}$	$\begin{array}{r}12\frac{1}{2}\\ \underline{7\frac{1}{6}}\end{array}$
$\begin{array}{r}3\frac{1}{2}\\ \underline{2\frac{1}{4}}\end{array}$	$\begin{array}{r}5\frac{1}{6}\\ \underline{2\frac{1}{2}}\end{array}$	$\begin{array}{r}7\frac{3}{8}\\ \underline{3\frac{1}{4}}\end{array}$	$\begin{array}{r}6\frac{1}{4}\\ \underline{4\frac{7}{8}}\end{array}$	$\begin{array}{r}9\frac{1}{8}\\ \underline{3\frac{1}{2}}\end{array}$
$\begin{array}{r}6\frac{1}{6}\\ \underline{3\frac{1}{3}}\end{array}$	$\begin{array}{r}4\frac{5}{8}\\ \underline{1\frac{3}{4}}\end{array}$	$\begin{array}{r}3\frac{1}{2}\\ \underline{1\frac{5}{8}}\end{array}$	$\begin{array}{r}9\frac{3}{4}\\ \underline{5\frac{1}{8}}\end{array}$	$\begin{array}{r}7\frac{5}{6}\\ \underline{4\frac{2}{3}}\end{array}$
$\begin{array}{r}6\frac{1}{8}\\ \underline{2\frac{1}{4}}\end{array}$	$\begin{array}{r}4\frac{1}{4}\\ \underline{2\frac{3}{8}}\end{array}$	$\begin{array}{r}7\frac{1}{2}\\ \underline{2\frac{3}{4}}\end{array}$	$\begin{array}{r}8\frac{3}{8}\\ \underline{4\frac{1}{2}}\end{array}$	$\begin{array}{r}11\frac{7}{10}\\ \underline{4\frac{1}{2}}\end{array}$
$\begin{array}{r}12\frac{1}{2}\\ \underline{8\frac{1}{12}}\end{array}$	$\begin{array}{r}6\frac{1}{4}\\ \underline{3\frac{5}{8}}\end{array}$	$\begin{array}{r}14\frac{1}{2}\\ \underline{10\frac{1}{8}}\end{array}$	$\begin{array}{r}11\frac{1}{2}\\ \underline{8\frac{5}{6}}\end{array}$	$\begin{array}{r}9\frac{3}{4}\\ \underline{5\frac{3}{8}}\end{array}$
$\begin{array}{r}6\frac{7}{8}\\ \underline{3\frac{1}{2}}\end{array}$	$\begin{array}{r}10\frac{2}{3}\\ \underline{5\frac{1}{6}}\end{array}$	$\begin{array}{r}14\frac{7}{8}\\ \underline{8\frac{3}{4}}\end{array}$	$\begin{array}{r}5\frac{5}{6}\\ \underline{1\frac{1}{2}}\end{array}$	$\begin{array}{r}7\frac{1}{4}\\ \underline{3\frac{1}{8}}\end{array}$

Denominators without common factors. In the practice examples of Set 10, the least common denominator is the product of the two given denominators. The

skill which the pupils have already developed in the addition of such fractions should carry over to the 25 examples of this set. While pupils are working on examples of this kind, the teacher should be on the alert for evidences of difficulty in borrowing, for tendencies to perform the wrong operation, for difficulty in changing the fractions to common denominator form, for errors in computation, and for other evidences of misunderstanding or the lack of understanding. Of course, the teacher will supplement the set with examples requiring the subtraction of fractions from fractions, of fractions from integers, of fractions from mixed numbers, of integers from mixed numbers, and of mixed numbers from integers.

PRACTICE EXAMPLES IN FRACTIONS. SET 10

Common Denominator by Multiplication

$\begin{array}{r}7\frac{1}{2}\\ \underline{3\frac{1}{3}}\end{array}$	$\begin{array}{r}4\frac{2}{3}\\ \underline{1\frac{1}{2}}\end{array}$	$\begin{array}{r}3\frac{1}{3}\\ \underline{1\frac{1}{4}}\end{array}$	$\begin{array}{r}7\frac{1}{4}\\ \underline{2\frac{2}{3}}\end{array}$	$\begin{array}{r}6\frac{1}{3}\\ \underline{3\frac{1}{8}}\end{array}$
$\begin{array}{r}6\frac{2}{3}\\ \underline{4\frac{2}{5}}\end{array}$	$\begin{array}{r}3\frac{1}{5}\\ \underline{1\frac{1}{2}}\end{array}$	$\begin{array}{r}8\frac{1}{4}\\ \underline{5\frac{1}{3}}\end{array}$	$\begin{array}{r}9\frac{1}{2}\\ \underline{4\frac{4}{5}}\end{array}$	$\begin{array}{r}6\frac{2}{5}\\ \underline{3\frac{2}{3}}\end{array}$
$\begin{array}{r}5\frac{5}{8}\\ \underline{2\frac{1}{3}}\end{array}$	$\begin{array}{r}7\frac{2}{3}\\ \underline{4\frac{1}{4}}\end{array}$	$\begin{array}{r}6\frac{1}{2}\\ \underline{1\frac{3}{5}}\end{array}$	$\begin{array}{r}7\frac{1}{3}\\ \underline{3\frac{1}{5}}\end{array}$	$\begin{array}{r}5\frac{1}{2}\\ \underline{2\frac{2}{3}}\end{array}$
$\begin{array}{r}14\frac{3}{4}\\ \underline{10\frac{2}{3}}\end{array}$	$\begin{array}{r}11\frac{3}{8}\\ \underline{6\frac{1}{3}}\end{array}$	$\begin{array}{r}7\frac{2}{5}\\ \underline{3\frac{1}{2}}\end{array}$	$\begin{array}{r}5\frac{3}{8}\\ \underline{1\frac{2}{3}}\end{array}$	$\begin{array}{r}4\frac{1}{3}\\ \underline{2\frac{1}{2}}\end{array}$
$\begin{array}{r}8\frac{2}{3}\\ \underline{5\frac{3}{4}}\end{array}$	$\begin{array}{r}6\frac{3}{4}\\ \underline{2\frac{1}{3}}\end{array}$	$\begin{array}{r}10\frac{1}{2}\\ \underline{4\frac{1}{5}}\end{array}$	$\begin{array}{r}9\frac{2}{3}\\ \underline{6\frac{3}{8}}\end{array}$	$\begin{array}{r}7\frac{1}{3}\\ \underline{1\frac{3}{4}}\end{array}$

Common denominator by inspection. Examples of this type are found in Set 11. The 30 examples of this set

provide for considerable variety but are confined largely to combinations of fourths and sixths, and sixths and eighths. A few other combinations are included that the pupil may see that the method which he is learning is of general application. Again, the teacher will supplement the set by providing examples in which the numbers are not all mixed numbers.

Practice Examples in Fractions. Set 11

Common Denominator by Inspection

$3\frac{1}{4}$ $1\frac{5}{6}$	$6\frac{1}{6}$ $1\frac{1}{4}$	$7\frac{1}{6}$ $2\frac{1}{8}$	$5\frac{3}{8}$ $1\frac{5}{6}$	$4\frac{1}{6}$ $1\frac{5}{8}$
$9\frac{2}{9}$ $4\frac{1}{6}$	$6\frac{1}{8}$ $2\frac{1}{6}$	$5\frac{5}{6}$ $2\frac{3}{4}$	$7\frac{3}{4}$ $4\frac{3}{10}$	$8\frac{5}{6}$ $3\frac{7}{8}$
$4\frac{1}{8}$ $1\frac{5}{6}$	$3\frac{1}{6}$ $1\frac{1}{10}$	$7\frac{3}{4}$ $4\frac{5}{6}$	$6\frac{1}{6}$ $1\frac{3}{4}$	$14\frac{7}{8}$ $5\frac{1}{6}$
$4\frac{5}{6}$ $2\frac{1}{8}$	$8\frac{3}{8}$ $3\frac{7}{10}$	$5\frac{5}{8}$ $2\frac{5}{6}$	$4\frac{3}{4}$ $2\frac{1}{6}$	$6\frac{1}{6}$ $2\frac{3}{8}$
$7\frac{3}{8}$ $4\frac{1}{6}$	$5\frac{5}{6}$ $2\frac{5}{8}$	$7\frac{1}{4}$ $3\frac{1}{6}$	$9\frac{7}{8}$ $3\frac{5}{12}$	$8\frac{1}{6}$ $2\frac{7}{8}$
$7\frac{5}{8}$ $2\frac{1}{6}$	$12\frac{5}{6}$ $10\frac{3}{8}$	$9\frac{5}{6}$ $5\frac{1}{4}$	$3\frac{1}{6}$ $1\frac{1}{9}$	$10\frac{7}{8}$ $6\frac{5}{6}$

The sequence of topics. We have recognized the same major types of examples in addition and subtraction of fractions. In each, we have had similar fractions and unlike fractions and in each, we have divided the latter into three distinct groups. We have also seen that teach-

ing children to subtract fractions is in many respects similar to teaching them to add fractions.

It seems best, then, to teach pupils to subtract any given type of examples immediately after they have learned to add that type. After pupils have become proficient in adding similar fractions they may subtract similar fractions. Then they may learn to add unlike fractions with the least common denominator given and follow with subtraction of fractions of the same type. After that, practice will be given on the remaining two types of unlike fractions, first in addition and then in subtraction.

QUESTIONS AND REVIEW EXERCISES

1. What are the probable effects of undertaking to teach pupils to add and subtract fractions before they have had a sufficient background of experience with fractions to enable them to know what fractions are and what they mean?

2. Is a teacher responsible for making up the deficiencies in a pupil's preparation which are due to neglect on the part of another teacher?

3. Why do pupils become confused about the signs, $+$ and $\times$?

4. Is adding denominators a natural mistake for a pupil to make? If a pupil says that $\frac{1}{4}$ and $\frac{1}{4}$ make $\frac{2}{8}$, what, probably, is his fundamental difficulty?

5. What reasons are there for doubting the validity of error studies which have been made from test papers alone? Do you think that such error studies made by some persons are much more valid than those made by others? What additional sources of information would you use in analyzing pupil errors?

6. Do you believe that some errors in adding fractions

are more serious than others? Should the teacher treat all errors as of equal importance?

7. State the major classes of examples in the addition of fractions which are recognized in this chapter. What five types of examples in the addition of like fractions were recognized?

8. Describe an introductory lesson which you would use in teaching children to add like fractions.

9. Should examples in the addition of fractions be presented in both the equation and the column form of statement? Which form should predominate?

10. Should the major portion of the pupil's practice material in addition of fractions consist of fractions alone or mixed numbers? Why?

11. Why are there no halves in the examples of Set 3?

12. How would you teach pupils to change improper fractions to mixed numbers? When should the rule, "Divide the numerator by the denominator" be learned?

13. Describe a lesson in which you would teach pupils to carry in addition of fractions.

14. Why is it advisable to encourage pupils to do "in their heads" much of the work of changing fractions to common denominator form, changing improper fractions to mixed numbers, and reducing fractions to lowest terms?

15. Should pupils have had sufficient experience with fractions to know that $\frac{1}{2}=\frac{2}{4}$ before they approach the addition of such fractions?

16. How and when should pupils learn the principle that both the numerator and the denominator of a fraction can be multiplied by the same number without changing the value of the fraction?

17. When is the product of two or more numbers their least common multiple? Is the least common multiple of a group of denominators the least common denominator of those fractions?

18. How would you teach pupils to find the least common denominator when the denominators have one or more factors in common?

19. What is meant by the statement that finding the least common denominator is a matter of convenience and not a matter of vital importance?

20. Is it reasonable to expect pupils to learn that $\frac{1}{2}$ and $\frac{1}{3}$ are $\frac{5}{6}$ as they learn that 6 and 7 are 13?

21. What is the commonest error which was found by Brueckner in the subtraction of fractions?

22. What method of subtraction is assumed in connection with the discussion of subtraction of similar fractions with borrowing? What changes should be made in this explanation if the pupils have learned the take-away-carry method? The addition-carry method?

23. Why does this chapter devote less space to the discussion of the subtraction of unlike fractions than to the discussion of the addition of unlike fractions?

24. What is meant by finding common denominators by inspection?

25. How much work in the addition of fractions should be completed before the subtraction of fractions is begun? What should be the teaching order for the remaining work in addition and subtraction of fractions?

26. How could you use a foot rule to illustrate the addition of such fractions as $\frac{1}{2}$ and $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{1}{4}$, or $\frac{1}{3}$ and $\frac{1}{4}$? Can you illustrate the subtraction of fractions with a foot rule also?

CHAPTER TEST

For each of these statements, select the best answer. A key will be found on page 533.

1. Pupils frequently confuse addition of fractions, when the examples are expressed in the equation form, with (1) subtraction (2) multiplication (3) division.

2. The table showing the per cent of pupils scoring zero on a test in the addition of fractions showed the largest per cent of zero scores in (1) the sixth grade (2) the seventh grade (3) the eighth grade.

3. The pupil who got $\frac{2}{5}$ as the answer to $\frac{2}{3}+\frac{3}{5}$ probably (1) subtracted (2) multiplied (3) divided.

4. It is desirable that error studies be based upon (1) test papers (2) interviews (3) both test papers and interviews.

5. Pupils who cancel in examples in the addition of fractions probably (1) do not understand addition of fractions (2) have just been taught cancelation (3) are trying to divide the fractions.

6. The majority of examples in the addition of fractions should consist of (1) mixed numbers only (2) fractions only (3) both fractions and mixed numbers.

7. The most serious condition found by Brueckner in his study of errors in the addition of fractions was (1) errors in reducing to lowest terms (2) lack of understanding of the process (3) errors in computation.

8. The first example in addition of like fractions may be

$$(1)\ \begin{array}{r} 3\frac{1}{3} \\ \underline{2\frac{1}{3}} \end{array} \qquad (2)\ \begin{array}{r} 1\frac{1}{4} \\ \underline{3\frac{3}{4}} \end{array} \qquad (3)\ \begin{array}{r} 1\frac{1}{2} \\ \underline{2\frac{1}{2}} \end{array}.$$

9. Fractions having different numbers as denominators are called (1) similar fractions (2) proper fractions (3) unlike fractions.

10. The first set of practice examples in addition of fractions should not include (1) halves (2) fourths (3) sixths.

11. An improper fraction is one in which (1) the numerator is equal to the denominator (2) the numerator is greater than the denominator (3) the numerator is equal to or greater than the denominator.

12. In ordinary business and social affairs improper

fractions occur (1) more often than mixed numbers (2) less often than mixed numbers (3) just as often as mixed numbers.

13. In adding mixed numbers, well-taught pupils learn eventually to (1) put all of their work on paper (2) put some of their work on paper (3) put none of their work on paper.

14. The value of a fraction is unchanged if both the numerator and the denominator are (1) multiplied by the same number (2) increased by the same number (3) decreased by the same number.

15. Pupils should eventually memorize the equivalence of (1) $\frac{1}{3}$ and $\frac{3}{9}$ (2) $\frac{1}{2}$ and $\frac{7}{14}$ (3) $\frac{1}{5}$ and $\frac{2}{10}$.

16. The least common denominator is the product of the given denominators if these denominators (1) have no factors in common (2) have one or more factors in common (3) are identical.

17. To add fractions whose denominators have common factors pupils should (1) factor the denominators (2) find the product of the denominators (3) try successive multiples of the largest denominator.

18. Finding the least common denominator is (1) undesirable (2) mathematically necessary (3) convenient.

19. The commonest error found by Brueckner in subtraction of fractions was (1) error in reducing to lowest terms (2) error in borrowing (3) error in computation.

20. The number of major classes of examples in subtraction of fractions is (1) equal to (2) less than (3) greater than the number of major classes in addition of fractions.

21. Subtraction of like fractions should be taught (1) before addition of like fractions (2) just after addition of like fractions (3) after addition of fractions has been completed.

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2. Brueckner, Leo J. "Analysis of Errors in Fractions." *Elementary School Journal*, XXVIII: 760-770, June, 1928. Reports a detailed analysis of 21,065 errors in the four operations in fractions made by 200 fifth- and sixth-grade pupils.

3. Brueckner, Leo J. *Diagnostic and Remedial Teaching in Arithmetic*. Philadelphia: The John C. Winston Company, 1930, 341 pp. Chapter V reports the analyses of errors in fractions reported in Reference 2, an analysis of specific abilities in fractions, and brief suggestions for teaching.

4. Brueckner, Leo J. "A Critical Evaluation of Methods of Analyzing Practice in Fractions." National Society for the Study of Education, *Twenty-Ninth Yearbook*. Bloomington, Illinois: Public School Publishing Company, 1930, pp. 525-534. Criticizes Knight's "unit skills" analysis.

5. Karpinski, Louis Charles. *The History of Arithmetic*. Chicago: Rand McNally and Company, 1925, 200 pp. Chapter V presents an interesting historical discussion of fractions.

6. Klapper, Paul. *The Teaching of Arithmetic*. New York: D. Appleton-Century Company, 1934, 525 pp. A brief but well-organized discussion of addition and subtraction of fractions will be found on pages 350-360.

7. Knight, F. B. "Some Considerations of Method."

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8. Morton, R. L. "An Analysis of Pupils' Errors in Fractions." *Journal of Educational Research,* IX: 117-125, February, 1924. Presents an analysis of the errors of one class in the four operations with fractions.

9. Roantree, William F. and Taylor, Mary S. *An Arithmetic for Teachers.* New York: The Macmillan Company, 1932, 523 pp. The teacher's knowledge of fractions is discussed on pages 121-146. Methods of teaching the addition and subtraction of fractions are presented on pages 151-162.

10. Taylor, E. H. *Arithmetic for Teacher-Training Classes.* New York: Henry Holt and Company, 1937, 432 pp. Material for the teacher's use is contained in Chapter VII.

11. Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937, 591 pp. The addition and subtraction of fractions is treated briefly on pages 390-395.

12. Wildeman, Edward. *The Teaching of Fractions.* Chicago: The Plymouth Press, 1923, 145 pp. Suggestions on the addition and subtraction of fractions will be found in Chapter III, pages 54-77.

CHAPTER 6

MULTIPLICATION AND DIVISION WITH FRACTIONS

When pupils are given tests on the four fundamental operations with fractions, it is not unusual for them to make better scores, on the average, on the test in multiplication than on tests in the other three operations. This is especially true if tests include fractions only, rather than mixed numbers, and if examples appear in the equation form. This leads to the conclusion that multiplication seems to be the easiest of the operations with fractions. The reason is obvious. In multiplication, pupils do just what they have learned to do in multiplication with integers; they react to the familiar × sign by multiplying numerators for a new numerator and denominators for a new denominator. Now, pupils who have little understanding of fractions and who have not mastered any of the operations with them also add numerators and denominators in attempting to solve examples in addition and subtract them when trying to solve subtraction examples but correct answers are never obtained in this way for such examples. It just happens that a similar procedure, followed with no greater understanding of the process, always yields correct results in multiplication if there are no errors in computation and if results are properly reduced. It seems a doubtful conclusion, then, that multiplication is really appreciably easier than addition or subtraction.

Division, on the other hand, is definitely the hardest of the operations with fractions to learn if "learning"

the operation implies an understanding of the steps in the solution. In division, the pupils do a new and heretofore unheard-of thing to the divisor; and then they cease thinking about division and multiply instead. We shall attempt to show, however, that division need be neither so meaningless nor so difficult as it is made in many schools. Division is difficult and good teaching will not make it easy. But it can be made easier. Certainly, it can be made more meaningful than it is to many pupils.

THE MULTIPLICATION OF FRACTIONS

Should multiplication be taught first? Convinced that multiplication is the easiest of the operations with fractions, some students of the teaching of arithmetic recommend that multiplication be taught before addition or subtraction. A few schools have tried this plan and have reported that the results seemed to be satisfactory. It should be remembered, however, that the success of pupils in multiplication is probably due to good fortune as well as to the ease of the operation with some examples. Furthermore, if multiplication is presented first and the pupils learn that they multiply numerators for a new numerator and denominators for a new denominator, there may be a still stronger tendency to add denominators in addition and to subtract them in subtraction. It seems best, then, to follow the plan outlined by the textbook and the course of study until experimental evidence justifying a change is produced.

Pupils' errors in multiplication. Brueckner found that far fewer errors were made in multiplication of

fractions than in addition or subtraction.[1] Whereas, the 200 pupils whose test papers he analyzed made 6,202 errors in addition and 7,511 errors in subtraction, they made only 2,477 errors in multiplication. The author also has found fewer errors in multiplication than in addition or subtraction.[2]

The commonest error found by Brueckner was errors in computation. He listed 712 of these, or 28.7 per cent of the total. Next in order of frequency came errors indicating a failure to understand the process of multiplication; there were 429 of these, or 17.3 per cent of the total. Next came failure to reduce to lowest terms or difficulty in reducing when it was attempted; in this class, there were 428 errors, or 17.3 per cent of the total. There were 218 cases of failure to change improper fractions to mixed numbers; this was 8.8 per cent of all the errors made. There were also occasional errors in copying figures, in changing mixed numbers to improper fractions, and in cancelation. In 214 cases, the difficulty was unknown and in 280 cases, the example was omitted. This last group can hardly be classified as errors.

In the preceding chapter, it was suggested that there is considerable variation from school to school and from grade to grade in the errors made and in the proportion of each. For example, Brueckner found that 33.1 per cent of the errors made in Grade VIB were errors in computation while only 24.4 per cent of the errors made in Grade VIA belonged in this class. On the other

[1] Brueckner, Leo J. *Diagnostic and Remedial Teaching in Arithmetic.* Philadelphia: The John C. Winston Company, 1930, pp. 211-213.

[2] Morton, R. L. "An Analysis of Pupils' Errors in Fractions." *Journal of Educational Research*, IX: 117-125, February, 1924.

hand, errors that indicated a failure to understand the process of multiplication constituted 13.3 per cent of the total in Grade VIB but 21.3 per cent of the total in Grade VIA. The total number of errors made was almost exactly the same in Grade VIB and Grade VIA.[3] In other words, the teacher who makes an analysis of the errors made by her pupils should not be surprised if the proportion in each of these groups differs considerably from the proportions found by Brueckner or by other teachers. It should be remembered, too, that Brueckner's analysis was made from test papers but that a more accurate and a more nearly complete analysis can be made by supplementing such an analysis with information secured through the use of the interview technique.

The author has found that if the test consists of examples containing simple fractions alone and they are presented in the equation form, there is sometimes a marked tendency to read the sign $\times$ as $+$ and to add the fractions instead of multiplying them.[4]

The class whose test papers were analyzed by the author made 69 errors in the three types, error in computation, failure to reduce answers to lowest terms (or cancel), and failure to reduce answers to mixed numbers. A few months later, the test was repeated with this class. This time, there were only 29 errors in these three groups. In the interval between the two tests, the teacher used a series of practice exercises in fractions—all four operations—but did not give special attention to the three types of errors under consideration, for the

[3] Brueckner, Leo J., *op. cit.*, p. 211.
[4] Morton, R. L., *loc. cit.*

analysis of errors had not been made at that time. There was abundant evidence that special attention to fractions and more careful teaching produces a marked improvement in the average performance of the pupils.

It is interesting to note that pupils who make good scores occasionally lapse. One, whose score was 20-18 (20 tried and 18 correct), added denominators in the two wrong examples, in one case after canceling. These were the examples: $\overset{1}{\cancel{3}}/8 \times 1/\underset{2}{\cancel{6}} = 1/10$; $1/2 \times 1/8 = 1/10$. It is probable that this pupil had been trying recently to overcome the habit of adding denominators or of confusing addition with multiplication. Myers shows that errors once made are likely to occur again and again.[5]

Brueckner and Elwell studied the validity of a diagnosis of errors in multiplication of fractions.[6] They concluded that a single example of a given type is likely to give an incorrect impression of what the individual pupils can do with examples of that type. Ordinarily, there should be at least three examples of a type for accurate individual diagnosis.

Types of examples in multiplication. It is possible to recognize a great many types of examples in multiplication with fractions. Brueckner lists 45 types.[7] Six of these 45 are types involving three factors instead of just

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[5] Myers, Garry Cleveland. *The Prevention and Correction of Errors in Arithmetic.* Chicago: The Plymouth Press, 1925, pp. 5-9.

[6] Brueckner, Leo J. and Elwell, Mary. "Reliability of Diagnosis of Error in Multiplication of Fractions." *Journal of Educational Research,* XXVI: 175-185, November, 1932.

[7] Brueckner, Leo J., *op. cit.,* pp. 186-191.

two and five others are types which were found to be especially difficult because they involve the use of large numbers. This leaves 34 types. These 34 types are classified in eight groups. The eight groups provide for all possible arrangements of integers, proper fractions, and mixed numbers. Thus we can have:

1. Proper fractions multiplied by integers.
2. Integers multiplied by proper fractions.
3. Mixed numbers multiplied by integers.
4. Integers multiplied by mixed numbers.
5. Proper fractions multiplied by proper fractions.
6. Mixed numbers multiplied by proper fractions.
7. Proper fractions multiplied by mixed numbers.
8. Mixed numbers multiplied by mixed numbers.

Subdivisions of these groups include those in which the product can be reduced to lower terms (or in which cancelation is possible), those in which the product is an integer, those in which the product is a mixed number, and various combinations of these.

For convenience, we can classify all examples in multiplication with fractions in three groups and then arrange for the necessary types in each group. These three groups are:

1. Fractions multiplied by integers.
2. Integers multiplied by fractions.
3. Fractions multiplied by fractions.

In each of these three groups, we shall understand "fractions" to include both proper fractions and mixed numbers. The first of these three groups would include the first and the third of the above eight groups; the second would include the second and the fourth of the eight

groups; and the third would include the fifth, the sixth, the seventh, and the eighth of the eight groups.

Multiplication of a fraction by an integer. This group seems to be the best group to use in giving pupils their first acquaintance with multiplication of fractions. The first examples probably should be cases of multiplying a proper fraction by an integer. We may begin with unit fractions and then move to other proper fractions. It hardly seems to be necessary to avoid examples which yield products that can be reduced to lower terms or changed to integers or mixed numbers since the pupils already have had much experience with such numbers. We may simply let the products occur as they will. We may consider in this major group, then, cases of:

(a) Multiplication of a unit fraction by an integer.
(b) Multiplication of any other proper fraction by an integer.
(c) Multiplication of a mixed number by an integer.

A problem, reflecting some phase of pupil experience, should be found to introduce the subject. The teacher may state a problem and develop the solution as follows:

TEACHER: "Henry drinks one-half of a pint of milk each morning for breakfast. How many pints does he drink in a week? Suppose that Henry drank 2 pints of milk each day. How many pints would he drink in 7 days?"

PUPIL: "14 pints."

TEACHER: "How did you get 14 pints?"

PUPIL: "Multiply 2 pints by 7."

TEACHER: "But Henry really drinks $\frac{1}{2}$ pint each day. How many half-pints will he drink in 7 days?"

PUPIL: "7 half-pints."

TEACHER: "We can write it this way: 7×1 half-pint $= 7$ half-pints; or this way, $7 \times \frac{1}{2}$ pint $= \frac{7}{2}$ pints. The second way is better. How many pints in $\frac{7}{2}$ pints?"

PUPIL: "$3\frac{1}{2}$ pints."

TEACHER: "Then we can write, $7 \times \frac{1}{2}$ pint $= \frac{7}{2}$ pints $= 3\frac{1}{2}$ pints, or, leaving out the word, 'pints,' $7 \times \frac{1}{2} = \frac{7}{2} = 3\frac{1}{2}$."

There should be several such illustrative problems with accompanying examples. Here is another:

TEACHER: "From Ruth's house to the grocery and back is just $\frac{1}{4}$ of a mile. In one week, Ruth walked to the grocery 12 times. How many miles did she walk? How do we find out?"

PUPIL: "Multiply $\frac{1}{4}$ by 12."

TEACHER: "We can write, 12×1 fourth $= 12$ fourths $= 3$, or $12 \times \frac{1}{4} = \frac{12}{4} = 3$."

In like manner, many other proper fractions can be used. These will include the following and others. Note that some of the products are reducible to integers and some to mixed numbers.

12×3 fourths $= 36$ fourths $= 9$	$12 \times \frac{3}{4} = \frac{36}{4} = 9$
6×1 third $= 6$ thirds $= 2$	$6 \times \frac{1}{3} = \frac{6}{3} = 2$
8×2 thirds $= 16$ thirds $= 5\frac{1}{3}$	$8 \times \frac{2}{3} = \frac{16}{3} = 5\frac{1}{3}$
7×1 fifth $= 7$ fifths $= 1\frac{2}{5}$	$7 \times \frac{1}{5} = \frac{7}{5} = 1\frac{2}{5}$
10×2 fifths $= 20$ fifths $= 4$	$10 \times \frac{2}{5} = \frac{20}{5} = 4$
12×3 eighths $= 36$ eighths $= 4\frac{1}{2}$	$12 \times \frac{3}{8} = \frac{36}{8} = 4\frac{1}{2}$

From these examples, there emerges gradually the general idea that, *To multiply a fraction by a whole number, multiply the numerator by the whole number and write the denominator under the product.* It is not to be recommended that the pupils memorize and recite this statement as a rule but they should understand it. If they understand it, they can state it in their own words and use it intelligently. Also, this principle or rule is not to be handed out to the pupils at the beginning of their work in multiplication with fractions but is to be discovered by them through simple examples which arise from meaningful situations.

Practice Examples in Fractions, Set 12, provide practice on the types of examples under consideration. In each of the first 16 examples of this set, the fraction is a unit fraction. Other proper fractions occur in the remaining 24 examples. Each of the fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$, occurs twice in this set, once with a multiplier which is a multiple of the denominator of the fraction and once with a multiplier which is not a multiple of the denominator of the fraction. A few sevenths, ninths, tenths, and twelfths are added.

Practice Examples in Fractions. Set 12

Multiplication of a Fraction by an Integer

$6 \times \frac{1}{2} =$	$9 \times \frac{1}{3} =$	$7 \times \frac{1}{4} =$	$10 \times \frac{1}{5} =$
$4 \times \frac{1}{6} =$	$9 \times \frac{1}{2} =$	$12 \times \frac{1}{8} =$	$12 \times \frac{1}{6} =$
$8 \times \frac{1}{4} =$	$5 \times \frac{1}{3} =$	$6 \times \frac{1}{5} =$	$16 \times \frac{1}{8} =$
$7 \times \frac{1}{7} =$	$4 \times \frac{1}{10} =$	$6 \times \frac{1}{12} =$	$13 \times \frac{1}{9} =$
$8 \times \frac{3}{8} =$	$7 \times \frac{5}{6} =$	$40 \times \frac{5}{8} =$	$10 \times \frac{3}{4} =$
$15 \times \frac{2}{3} =$	$20 \times \frac{7}{8} =$	$12 \times \frac{3}{8} =$	$6 \times \frac{5}{6} =$
$15 \times \frac{2}{9} =$	$10 \times \frac{2}{5} =$	$13 \times \frac{3}{5} =$	$9 \times \frac{5}{8} =$

$12 \times \frac{2}{7} =$	$20 \times \frac{3}{4} =$	$11 \times \frac{2}{5} =$	$16 \times \frac{7}{8} =$
$5 \times \frac{4}{5} =$	$25 \times \frac{3}{10} =$	$3 \times \frac{4}{5} =$	$8 \times \frac{7}{12} =$
$8 \times \frac{2}{3} =$	$15 \times \frac{3}{5} =$	$16 \times \frac{5}{12} =$	$20 \times \frac{7}{10} =$

In solving such an example as $12 \times \frac{3}{8}$, most adults probably would cancel, obtaining $\frac{9}{2}$ as the product, rather than multiply and then reduce the product, $\frac{36}{8}$ to lowest terms. Children, too, should be taught to cancel for canceling reduces the size of the numbers with which they must deal and thereby makes the computations easier. But principles of good teaching demand that, so far as possible, but one new thing be introduced at a time. It seems best, then, to have the pupils become proficient in multiplying a fraction by an integer, reducing the product to lowest terms and to a mixed number, and then to introduce canceling as a short cut. Of course, canceling is nothing more than reducing the product to lowest terms before multiplying instead of after multiplying. We shall have more to say about canceling at a later point in the chapter.

The examples of Set 12 do not include cases of multiplying a mixed number by an integer. In examples of this type, the multiplicand can be changed to an improper fraction and the solution can be carried out in the equation form, or the form which the pupils have learned to use in multiplying integers may be employed. That is, if $3\frac{1}{2}$ is to be multiplied by 5, the work may appear in either of these forms. Both forms

$$5 \times 3\frac{1}{2} = 5 \times \frac{7}{2} = \frac{35}{2} = 17\frac{1}{2}$$

$$\begin{array}{r} 3\frac{1}{2} \\ \underline{5} \\ 2\frac{1}{2} \\ \underline{15} \\ 17\frac{1}{2} \end{array}$$

may be used. At first, the equation form is easier if the pupils have already had experience in multiplying a proper fraction by an integer. But, eventually, the second form should be the

form which is ordinarily used for it is brief, concise, and convenient.

In learning to use the second form, the pupil will see that he first multiplies ½ by 5 and then that he multiplies 3 by 5. The two products are then added to find the product of 3½ multiplied by 5. This is a little like adding partial products in multiplication of integers. Some teachers prefer to have pupils multiply 3 by 5 *before* multiplying ½ by 5. The work would appear as shown. Since the pupils are accustomed to beginning with the right-hand figures when the multiplicand has two or more digits, the first form shown seems to be better. However, the point does not seem to be one of much importance.

$$\begin{array}{r} 3\frac{1}{2} \\ \underline{5} \\ 15 \\ \underline{2\frac{1}{2}} \\ 17\frac{1}{2} \end{array}$$

The examples of Set 12 can easily be changed so as to provide practice in multiplying a mixed number by an integer. One simply affixes an integer to the fraction in each multiplicand and changes the multiplier if he thinks best. The first five examples in the set, then, might appear as follows:

$$\begin{array}{l} 5\frac{1}{2} \\ \underline{6\phantom{\frac{1}{2}}} \end{array} \qquad \begin{array}{l} 7\frac{1}{3} \\ \underline{9\phantom{\frac{1}{3}}} \end{array} \qquad \begin{array}{l} 4\frac{1}{4} \\ \underline{7\phantom{\frac{1}{4}}} \end{array} \qquad \begin{array}{l} 3\frac{1}{5} \\ \underline{10\phantom{\frac{1}{5}}} \end{array} \qquad \begin{array}{l} 8\frac{1}{6} \\ \underline{4\phantom{\frac{1}{6}}} \end{array}$$

Multiplication of an integer by a fraction. Arithmetically, multiplying an integer by a fraction is almost the same as multiplying a fraction by an integer. The fraction is a part of the multiplier in the one case and a part of the multiplicand in the other. But pupils and others will ordinarily use as multiplier the number which it is the more convenient to use as such, regardless of the logic of the situation.

There is frequently some difference in our actual experiences, however, between multiplying a fraction by an integer and multiplying an integer by a fraction. Consider these two problems.

1. Dick made 4 trips to the farm on his bicycle. He rode 6½ miles on each trip. How many miles did he ride?
2. Mrs. Johnson bought 2½ bushels of peaches at $2 per bushel. How much did they cost?

The first is clearly a case of multiplying 6½, the number of miles, by 4 while the second is a case of multiplying 2, the number of dollars, by 2½. The work would normally appear as follows:

$$
\begin{array}{r}
6\tfrac{1}{2} \\
\underline{4\phantom{\tfrac{1}{2}}} \\
2\phantom{\tfrac{1}{2}} \\
\underline{24\phantom{\tfrac{1}{2}}} \\
26\phantom{\tfrac{1}{2}}
\end{array}
\qquad\qquad
\begin{array}{l}
2 \\
\underline{2\tfrac{1}{2}} \\
1 \\
\underline{4\phantom{\tfrac{1}{2}}} \\
5
\end{array}
$$

This brings us back to the question of abstract and concrete numbers, a topic which was discussed in Chapter 3. (See pages 167-170.) We may repeat here that for children in the intermediate grades the distinction is not an important one. As a matter of convenience, we ordinarily use as multiplier the number having the fewer digits, rather than try to follow the rule about the product being like the multiplicand and the multiplier being an abstract number. Suppose, for example, that one wishes to find the cost of a gross (144) of an article at $1.25 each. The solution might appear as the first of the following but the second is easier and prob-

ably would be used by the majority of persons. The pupil does not hesitate to place the $ sign before the

$$\begin{array}{r} \$1.25 \\ \underline{144} \\ 500 \\ 500 \\ \underline{125} \\ \$180.00 \end{array} \qquad\qquad \begin{array}{r} 144 \\ \underline{1\tfrac{1}{4}} \\ 36 \\ \underline{144} \\ \$180 \end{array}$$

product in the second solution for he knows from his understanding of the conditions of the problem that his answer is dollars.

The teacher will do well to give practice on both the types under discussion. After the pupils have had sufficient practice in multiplying a fraction by an integer to learn the procedure rather well, they may be given practice in multiplying an integer by a fraction. Thereafter, the two types may occur miscellaneously in the material provided for practice.

The best approach to examples in which an integer is multiplied by a fraction seems to be through the use of problems requiring that an integer be multiplied by a proper fraction and in solving which the "of" form of statement would be used. The subject may be developed as follows:

TEACHER: "If chocolate creams are selling for 60 cents a pound, what will $\frac{1}{2}$ pound cost? How do we find out?"

PUPIL: "Find $\frac{1}{2}$ of 60."

TEACHER: "We can write it this way: $\frac{1}{2}$ of 60 cents = 30 cents. But finding $\frac{1}{2}$ of 60 cents is the same as multiplying 60 cents by $\frac{1}{2}$. Then, $\frac{1}{2}$ of $60 = \frac{1}{2} \times 60 = 30$."

Other illustrations will follow. Some of the examples will yield products which are integers while some will yield products which are mixed numbers. The pupil will see that *multiplying a whole number by a fraction* is very similar to multiplying a fraction by a whole number. In each case, he discovers that he finds the product of the numerator of the fraction and the whole number and writes this product over the denominator of the fraction.

Other proper fractions will then be used as multipliers. At first, these will be solved just as the examples containing unit fractions were solved. Thus, the pupil will write,

$$\tfrac{3}{4} \text{ of } 16 = \tfrac{3}{4} \times 16 = \tfrac{48}{4} = 12.$$

But such an example may also be solved by using a bit of analysis. The pupil learns that to find $\frac{3}{4}$ of 16 he can first find $\frac{1}{4}$ of 16 and then multiply the result by 3. This will come along a little later, however. This procedure leads directly into cancelation and should be learned as a short cut after the regular procedure has been learned.

When pupils have learned to multiply integers by unit fractions and by other proper fractions, it is but a short step to the multiplication of integers by mixed numbers. The elements have all been learned and the pupils simply put them together. By this time, they should be able to multiply integers by common fractions "in their heads" and to reduce the products to lowest terms and to mixed number form without any writing.

$$\begin{array}{r} 7 \\ \underline{2\tfrac{2}{3}} \\ 4\tfrac{2}{3} \\ \underline{14\phantom{\tfrac{2}{3}}} \\ 18\tfrac{2}{3} \end{array}$$

All that is written would be that which is shown here where 7 is multiplied by $2\frac{2}{3}$.

Set 13 of our practice examples in fractions contains 40 examples requiring multiplication of an integer by a mixed number. In contrast with the examples of Set 12, the multipliers of Set 13 appear as mixed numbers instead of as fractions alone and do not appear in the equation form of statement. Of course, the use of these examples will be preceded by the use of examples in which an integer is multiplied by a fraction alone. The teacher can easily modify the examples of Set 13 by eliminating the integer from the multiplier and can express them in the equation form. To conserve space, only the form appearing in Set 13 is included here.

Practice Examples in Fractions. Set 13

Multiplication of an Integer by a Mixed Number

8	12	12	4	10
$2\frac{1}{4}$	$3\frac{1}{3}$	$2\frac{1}{5}$	$3\frac{1}{2}$	$5\frac{1}{8}$
6	7	10	16	5
$5\frac{1}{6}$	$5\frac{1}{2}$	$8\frac{1}{6}$	$3\frac{1}{8}$	$2\frac{1}{3}$
7	11	14	8	14
$4\frac{1}{5}$	$6\frac{1}{4}$	$2\frac{1}{9}$	$3\frac{1}{12}$	$5\frac{1}{7}$
8	24	9	16	14
$3\frac{1}{10}$	$7\frac{5}{8}$	$4\frac{5}{6}$	$2\frac{3}{8}$	$1\frac{3}{4}$
20	6	18	12	14
$4\frac{3}{8}$	$2\frac{5}{6}$	$3\frac{7}{8}$	$2\frac{2}{3}$	$5\frac{3}{5}$
7	15	12	16	13
$2\frac{5}{8}$	$1\frac{2}{5}$	$4\frac{2}{9}$	$2\frac{3}{4}$	$9\frac{2}{7}$

9	32	15	5	4
$7\frac{2}{5}$	$5\frac{7}{8}$	$7\frac{3}{10}$	$3\frac{4}{5}$	$2\frac{7}{12}$
13	15	8	11	20
$2\frac{4}{5}$	$3\frac{3}{5}$	$4\frac{5}{12}$	$3\frac{2}{3}$	$5\frac{7}{10}$

Multiplication of a fraction by a fraction. The routine of multiplying a fraction by a fraction is easily learned. Pupils soon become proficient in multiplying numerators for a new numerator and denominators for a new denominator. Indeed, some teachers who are more interested in having pupils get right answers quickly than in having them understand the process of multiplication with fractions often have integers expressed in fraction form when the pupils are learning to multiply fractions by integers or integers by fractions. They make such an example as $4 \times \frac{2}{3}$ appear $\frac{4}{1} \times \frac{2}{3}$. This is a crutch which probably hinders more than it helps in developing a real understanding of the process although it may help more than it hinders in getting right answers without regard for whether or not the pupils understand the process. Learning the routine of multiplying a fraction by a fraction may be quite a different matter from understanding the procedure, and we want our pupils to be intelligent about these procedures. Skill in the fundamental operations of arithmetic is worth while but *skill with understanding is education.*

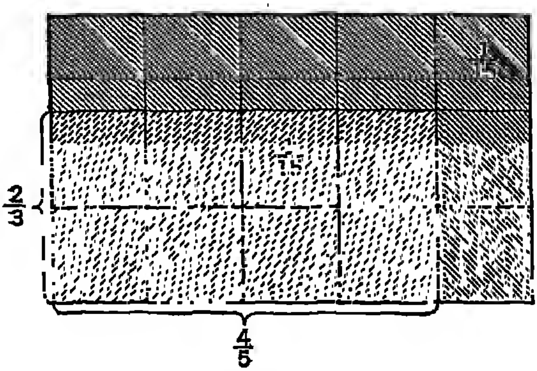
It is easy for the teacher to write on the blackboard an example such as, $\frac{1}{2} \times \frac{1}{2} =$, and say, "Now, boys and girls, to multiply $\frac{1}{2}$ by $\frac{1}{2}$, we just multiply the numerators for the numerator of our answer and we multiply the denominators for the denominator of our

answer," and proceed to do so. But here, as in many other places in the work of the elementary school, *telling is not teaching*. It is much better to begin with $\frac{1}{2}$ of $\frac{1}{2} =$, to show concretely that $\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$, and then to remind the pupils, if necessary, that "of" means "$\times$."

The following examples can be solved easily with the aid of diagrams such as those suggested in Chapter 4.

$\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$	$\frac{1}{2}$ of $\frac{2}{3} = \frac{1}{3}$	$\frac{1}{4}$ of $\frac{1}{3} = \frac{1}{12}$
$\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$	$\frac{2}{3}$ of $\frac{1}{2} = \frac{1}{3}$	$\frac{1}{3}$ of $\frac{3}{4} = \frac{1}{4}$
$\frac{1}{4}$ of $\frac{1}{2} = \frac{1}{8}$	$\frac{1}{2}$ of $\frac{3}{4} = \frac{3}{8}$	$\frac{3}{4}$ of $\frac{1}{3} = \frac{1}{4}$
$\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$	$\frac{3}{4}$ of $\frac{1}{2} = \frac{3}{8}$	$\frac{1}{2}$ of $\frac{1}{6} = \frac{1}{12}$
$\frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$	$\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{12}$	$\frac{1}{6}$ of $\frac{1}{2} = \frac{1}{12}$

The solution of such an example as $\frac{2}{3}$ of $\frac{4}{5}$ can be represented concretely as in the accompanying diagram.



The rectangle is divided into five vertical strips that are equal in width and area. Then, $\frac{4}{5}$ of the area of the rectangle is represented by four of these strips. The rectangle is divided also into three horizontal strips that are equal in width and area. Then $\frac{2}{3}$ of the area

of the rectangle is represented by two of these strips. It is seen that the rectangle is divided into fifteen small rectangles, each of which is $1/15$ of the original rectangle, and that $2/3$ of $4/5$ is represented by eight of these small rectangles. Thus, $2/3$ of $4/5$ is $8/15$.

After pupils have had the experience of working out a number of such examples with drawings, at least some of them probably will have discovered that in every case they can multiply the numerators to get the numerator of the product and multiply the denominators to get the denominator of the product but that sometimes the product must be reduced to lowest terms. Then, the general principle that the numerator of the answer is equal to the product of the numerators of the fractions which were to be multiplied and that the denominator of the answer is equal to the product of the denominators of the fractions which were to be multiplied is extended to include all examples in the multiplication of a fraction by a fraction whether the two fractions are connected by "of" or by "$\times$."

Such examples also show the pupils that the product is the same no matter in which order the fractions are stated. Of course, this is a well known fact since it has already been demonstrated many times in the multiplication of integers.

Examples for practice in multiplying a fraction by a fraction are given in Set 14. The 60 examples of this set are confined entirely to halves, thirds, fourths, fifths, sixths, and eighths. Some of the combinations of fractions given in this set will be of rare occurrence in ordinary affairs and some combinations which have not been included, such as $1/6 \times 1/8$, might have been given

a place. But it is believed that a sufficiently wide variety has been included to give pupils the practice they need and to enable them to cope with other situations involving the multiplication of a fraction by a fraction if they should be met.

Practice Examples in Fractions. Set 14
Multiplication of a Fraction by a Fraction

$\frac{1}{2} \times \frac{1}{4} =$	$\frac{1}{2} \times \frac{1}{2} =$	$\frac{1}{2}$ of $\frac{1}{3} =$	$\frac{1}{2} \times \frac{2}{3} =$
$\frac{1}{2}$ of $\frac{3}{4} =$	$\frac{1}{3}$ of $\frac{1}{4} =$	$\frac{3}{4} \times \frac{1}{3} =$	$\frac{1}{3} \times \frac{1}{3} =$
$\frac{1}{5} \times \frac{1}{2} =$	$\frac{1}{2}$ of $\frac{5}{6} =$	$\frac{1}{2} \times \frac{1}{8} =$	$\frac{1}{6} \times \frac{1}{3} =$
$\frac{2}{3}$ of $\frac{1}{4} =$	$\frac{1}{5} \times \frac{2}{3} =$	$\frac{4}{5} \times \frac{1}{2} =$	$\frac{1}{2} \times \frac{3}{8} =$
$\frac{1}{4} \times \frac{1}{4} =$	$\frac{1}{5} \times \frac{1}{4} =$	$\frac{3}{4}$ of $\frac{3}{4} =$	$\frac{1}{3}$ of $\frac{5}{6} =$
$\frac{1}{2}$ of $\frac{2}{5} =$	$\frac{1}{8} \times \frac{1}{3} =$	$\frac{3}{8} \times \frac{2}{3} =$	$\frac{1}{4}$ of $\frac{3}{8} =$
$\frac{5}{8}$ of $\frac{1}{2} =$	$\frac{1}{3} \times \frac{3}{5} =$	$\frac{2}{3}$ of $\frac{3}{4} =$	$\frac{1}{4} \times \frac{1}{6} =$
$\frac{3}{8} \times \frac{1}{3} =$	$\frac{1}{2} \times \frac{3}{5} =$	$\frac{2}{3}$ of $\frac{1}{6} =$	$\frac{1}{4}$ of $\frac{7}{8} =$
$\frac{1}{3}$ of $\frac{5}{8} =$	$\frac{1}{4}$ of $\frac{3}{4} =$	$\frac{3}{4} \times \frac{3}{8} =$	$\frac{3}{5} \times \frac{2}{3} =$
$\frac{1}{6}$ of $\frac{1}{2} =$	$\frac{7}{8} \times \frac{1}{3} =$	$\frac{2}{3}$ of $\frac{3}{5} =$	$\frac{3}{4} \times \frac{5}{8} =$
$\frac{3}{4} \times \frac{5}{6} =$	$\frac{1}{8}$ of $\frac{1}{4} =$	$\frac{1}{8}$ of $\frac{3}{4} =$	$\frac{2}{3} \times \frac{4}{5} =$
$\frac{1}{5} \times \frac{1}{3} =$	$\frac{2}{3}$ of $\frac{7}{8} =$	$\frac{5}{6} \times \frac{1}{4} =$	$\frac{3}{5}$ of $\frac{3}{4} =$
$\frac{2}{3}$ of $\frac{5}{6} =$	$\frac{1}{2}$ of $\frac{7}{8} =$	$\frac{1}{5}$ of $\frac{3}{4} =$	$\frac{1}{8} \times \frac{2}{3} =$
$\frac{2}{3} \times \frac{2}{3} =$	$\frac{3}{4} \times \frac{5}{8} =$	$\frac{1}{6}$ of $\frac{3}{4} =$	$\frac{1}{3} \times \frac{2}{3} =$
$\frac{2}{3} \times \frac{5}{8} =$	$\frac{1}{3} \times \frac{2}{5} =$	$\frac{2}{3}$ of $\frac{2}{5} =$	$\frac{3}{4}$ of $\frac{7}{8} =$

In multiplying a fraction by a mixed number or a mixed number by a fraction, it seems best to have pupils change the mixed number to an improper fraction and to use the equation form which they have used in multiplying a fraction by a fraction. A problem can be used and the solution developed as follows:

Teacher: "James measured a sheet of paper and found that it was just $8\frac{1}{2}$ inches wide. He wanted to cut it into four strips, all the same width. How wide should the strips be? How can we find out?"

PUPIL: "Find $\frac{1}{4}$ of $8\frac{1}{2}$."

TEACHER: "We have not learned how to multiply a mixed number by a fraction. This example would be easy if we changed $8\frac{1}{2}$ to an improper fraction. How many halves in 8?"

PUPIL: "16."

TEACHER: "And 1 half makes 17 halves. Then we can write, $\frac{1}{4}$ of $\frac{17}{2} = \frac{17}{8} = 2\frac{1}{8}$. So, each strip of paper should be $2\frac{1}{8}$ inches wide."

Changing mixed numbers to improper fractions. In solving examples in which one or more factors are mixed numbers, pupils should learn how to change a mixed number to an improper fraction. Simply telling them to multiply the whole number by the denominator of the fraction, to add the numerator to this product, and to write the denominator beneath the sum seldom leaves them with an understanding of the process. All but the duller pupils can see the reason for this procedure. The rule should come at the end of a series of meaningful experiences. If it is handed out ready-made at the beginning, it will be used by the pupils without understanding.

The method is simple and straightforward. Let us begin with halves. The pupils have learned that there are 2 halves in 1 apple, 4 halves in 2 apples, 6 halves in 3 apples, etc. Then they will readily see that in $2\frac{1}{2}$ apples there are 5 halves (written $\frac{5}{2}$) since there are 4 halves in 2 apples and 1 extra half, that in $1\frac{1}{2}$ apples there are 3 halves, that in $3\frac{1}{2}$ apples there are 7 halves, etc. Likewise, we can show concretely, if the pupils do not recall it, that there are 4 fourths in 1 whole thing, 8 fourths in 2 whole things, etc., and that in $2\frac{3}{4}$ there

8 fourths and 3 fourths or 11 fourths, written $1\frac{1}{4}$, etc. After it is apparent that the pupils understand this explanation, and can readily visualize the number of halves, thirds, and fourths in mixed numbers whose integers are small numbers, the short-cut procedure—multiply the whole number by the denominator, add in the numerator, and write the denominator under the sum—may be developed. The pupils should then be given practice in changing such mixed numbers as the following to improper fractions.

$3\frac{1}{2}$	$3\frac{1}{4}$	$1\frac{1}{3}$	$6\frac{2}{3}$	$4\frac{3}{4}$	$1\frac{2}{5}$
$2\frac{1}{2}$	$5\frac{1}{8}$	$4\frac{1}{5}$	$7\frac{1}{4}$	$3\frac{5}{8}$	$1\frac{3}{4}$
$9\frac{1}{6}$	$2\frac{3}{8}$	$7\frac{4}{5}$	$3\frac{5}{6}$	$6\frac{7}{8}$	$6\frac{3}{5}$

Set 15 contains examples for practice in multiplying a fraction by a mixed number and a mixed number by a fraction. When the pupil has converted the mixed number to an improper fraction, he will find that the procedure in solving the examples of this set is the same as that used with the examples of Set 14 except that his answers will often be improper fractions and will need to be changed to mixed numbers.

Practice Examples in Fractions. Set 15

Fractions and Mixed Numbers

$\frac{1}{2} \times 3\frac{1}{2} =$	$\frac{1}{4}$ of $2\frac{1}{2} =$	$\frac{1}{2}$ of $1\frac{1}{3} =$	$\frac{1}{2}$ of $6\frac{3}{4} =$
$\frac{1}{2} \times 7\frac{2}{3} =$	$\frac{3}{4} \times 3\frac{1}{3} =$	$\frac{1}{3}$ of $2\frac{1}{4} =$	$\frac{1}{3}$ of $1\frac{1}{8} =$
$6\frac{1}{2} \times \frac{1}{5} =$	$4\frac{3}{8} \times \frac{1}{2} =$	$2\frac{4}{5} \times \frac{1}{2} =$	$7\frac{1}{2} \times \frac{5}{6} =$
$1\frac{1}{6} \times \frac{2}{3} =$	$4\frac{2}{3} \times \frac{1}{4} =$	$2\frac{1}{2} \times \frac{1}{8} =$	$4\frac{1}{6} \times \frac{1}{3} =$
$3\frac{1}{4} \times \frac{3}{8} =$	$\frac{1}{4} \times 2\frac{1}{4} =$	$\frac{1}{2}$ of $4\frac{2}{5} =$	$\frac{1}{3}$ of $3\frac{5}{6} =$
$\frac{1}{5} \times 6\frac{1}{4} =$	$2\frac{3}{4} \times \frac{3}{4} =$	$7\frac{1}{8} \times \frac{1}{3} =$	$2\frac{2}{3} \times \frac{3}{8} =$
$7\frac{2}{3} \times \frac{3}{4} =$	$\frac{2}{3}$ of $4\frac{1}{6} =$	$\frac{1}{4}$ of $5\frac{7}{8} =$	$2\frac{1}{6} \times \frac{1}{4} =$
$1\frac{1}{5} \times \frac{3}{5} =$	$\frac{1}{2}$ of $8\frac{3}{5} =$	$\frac{3}{8} \times 10\frac{1}{3} =$	$\frac{5}{8} \times 9\frac{1}{2} =$

$1\frac{3}{4} \times \frac{5}{6} =$	$\frac{1}{8} \times 4\frac{3}{4} =$	$\frac{2}{3}$ of $4\frac{4}{5} =$	$\frac{1}{5} \times 7\frac{1}{3} =$
$3\frac{1}{8} \times \frac{2}{3} =$	$4\frac{3}{4} \times \frac{5}{8} =$	$2\frac{1}{3} \times \frac{2}{3} =$	$\frac{1}{3}$ of $6\frac{2}{5} =$

Multiplication of a mixed number by a mixed number. The most difficult of all the examples in the multiplication of fractions fall in this group. They are the most difficult because both the multiplier and the multiplicand must be changed to improper fractions (unless a more difficult process is used) and because larger numbers must be handled.

Two solution forms for multiplying a mixed number by a mixed number are found in use in the schools. The form shown at the left below is more commonly found and is undoubtedly easier, especially for children. The form at the right is also found, although not so frequently. In this form, it will be noted, four separate multiplications are involved. The pupil must multiply $\frac{1}{3}$ and 5 by $\frac{1}{4}$ and also $\frac{1}{3}$ and 5 by 3 and then he must add the four partial products. Pupils

$$3\frac{1}{4} \times 5\frac{1}{3} = \frac{13}{4} \times \frac{16}{3} = \frac{208}{12} = 17\frac{1}{3}$$

$$\begin{array}{r} 5\frac{1}{3} \\ \underline{3\frac{1}{4}} \\ \frac{1}{12} \\ 1\frac{1}{4} \\ 1 \\ \underline{15\phantom{\frac{1}{4}}} \\ 17\frac{1}{3} \end{array}$$

frequently make errors in following this procedure because of the complexity of the process. They find it much easier to reduce both mixed numbers to improper fractions and to proceed as in multiplying a fraction by a fraction. The more difficult form should not be used in the intermediate grades.

The teacher may develop the subject through the medium of a problem as follows:

TEACHER: "Mrs. Robinson bought $4\frac{1}{2}$ yards of material for a dress and paid \$3.50 per yard. How much did it cost? How shall we find out?"

PUPIL: "Multiply \$3.50 by $4\frac{1}{2}$."

TEACHER: "We can do this in two ways. Since \$3.50 is the same as \$$3\frac{1}{2}$, we can multiply $3\frac{1}{2}$ by $4\frac{1}{2}$. First, let us change these mixed numbers to improper fractions. Change $3\frac{1}{2}$ and $4\frac{1}{2}$ to improper fractions."

PUPIL: "$3\frac{1}{2} = \frac{7}{2}$ and $4\frac{1}{2} = \frac{9}{2}$."

TEACHER: "Then we can write,

$$4\frac{1}{2} \times 3\frac{1}{2} = \frac{9}{2} \times \frac{7}{2} = \frac{63}{4} = 15\frac{3}{4}$$

So, the material cost \$$15\frac{3}{4}$, or \$15.75. Now we can also multiply \$3.50 by $4\frac{1}{2}$ and prove our answer. Then, \$$15\frac{3}{4}$ or \$15.75, is correct."

\$3.50
$4\frac{1}{2}$
1.75
14.00
\$15.75

Examples for practice in multiplying a mixed number by a mixed number are given in Set 16. There are 35 examples in this set. Practice is limited to the fractions of more probable occurrence.

PRACTICE EXAMPLES IN FRACTIONS. SET 16

Multiplication of a Mixed Number by a Mixed Number

$6\frac{1}{4}$	$4\frac{1}{2}$	$3\frac{3}{8}$	$5\frac{1}{5}$	$6\frac{7}{8}$
$2\frac{1}{3}$	$7\frac{1}{4}$	$2\frac{1}{2}$	$1\frac{2}{3}$	$4\frac{1}{3}$
$3\frac{1}{2}$	$4\frac{1}{2}$	$7\frac{1}{6}$	$8\frac{3}{4}$	$7\frac{1}{2}$
$5\frac{1}{2}$	$1\frac{3}{4}$	$2\frac{1}{3}$	$2\frac{3}{4}$	$4\frac{1}{3}$
$6\frac{3}{8}$	$9\frac{1}{2}$	$10\frac{3}{5}$	$12\frac{1}{8}$	$5\frac{1}{3}$
$4\frac{1}{3}$	$2\frac{5}{8}$	$1\frac{1}{3}$	$5\frac{1}{4}$	$3\frac{1}{3}$
$16\frac{2}{3}$	$4\frac{1}{2}$	$7\frac{2}{3}$	$5\frac{1}{5}$	$6\frac{1}{6}$
$6\frac{2}{3}$	$3\frac{1}{8}$	$1\frac{1}{4}$	$3\frac{1}{3}$	$3\frac{1}{2}$
$5\frac{3}{5}$	$4\frac{3}{4}$	$6\frac{2}{3}$	$4\frac{1}{4}$	$7\frac{1}{3}$
$2\frac{1}{2}$	$7\frac{5}{8}$	$3\frac{1}{2}$	$3\frac{4}{5}$	$2\frac{5}{6}$

$8\frac{1}{2}$	$16\frac{1}{4}$	$1\frac{2}{5}$	$9\frac{1}{2}$	$3\frac{5}{6}$
$3\frac{7}{8}$	$4\frac{1}{4}$	$1\frac{2}{3}$	$4\frac{4}{5}$	$7\frac{1}{2}$
$9\frac{3}{4}$	$6\frac{3}{4}$	$4\frac{1}{2}$	$7\frac{2}{3}$	$6\frac{2}{5}$
$3\frac{2}{3}$	$6\frac{3}{5}$	$2\frac{1}{5}$	$3\frac{3}{8}$	$5\frac{1}{2}$

Cancelation. It has been suggested that cancelation should be taught as a short-cut method after the usual procedure of multiplying a fraction or a mixed number by an integer has been learned. Our discussion of cancelation at this point should not be taken to indicate that all the foregoing types of examples in the multiplication of fractions should be taught before the pupils are made acquainted with this shorter method. Both the teacher and the pupils should understand, however, that cancelation is not an essential but is a short method. It permits one to get as his product a fraction which does not have to be reduced to lower terms simply because it has already been reduced to lowest terms. It means a smaller dividend and a smaller divisor to manipulate in changing an improper fraction to a mixed number, if the product is not reduced to lowest terms before it is changed from an improper fraction to a mixed number.

Suppose that the following example has just been solved on the blackboard.

$$8 \times \frac{5}{6} = \frac{40}{6} = \frac{20}{3} = 6\frac{2}{3}.$$

Call attention to the fact that $\frac{40}{6}$ has been reduced to $\frac{20}{3}$ by dividing both the numerator and the denominator by 2. Then show that we could have gotten the same result by dividing the 8 and the 6 by 2 before multiplying, making the solution appear as follows:

$$\overset{4}{\cancel{8}} \times \frac{5}{\underset{3}{\cancel{6}}} = \frac{20}{3} = 6\frac{2}{3}.$$

If pupils see that $\frac{5}{6}$ is multiplied by 8 by multiplying the numerator, 5, by 8, they will see that 8 becomes a part of the numerator and that when they cancel, they reduce to lower terms before they multiply instead of after.

When a fraction is multiplied by a fraction, as in the example,

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2},$$

they should see that when they cancel, they are also reducing to lowest terms but that they are reducing before they multiply. Thus,

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{4}}} = \frac{1}{2}.$$

They reduced $\frac{6}{12}$ to lowest terms by dividing both 6 and 12 by 6. They could also have done this by dividing both 6 and 12 by 2 and then dividing both 3 and 6 by 3. When they canceled, they divided the numerator of the first fraction and the denominator of the second fraction by 2 and then they divided the denominator of the first and the numerator of the second by 3. They can do this before multiplying because the two numerators go together and the two denominators go together when they multiply. Thus, cancelation is a short-cut. Reducing to lowest terms is done before instead of after multiplying. Of course, the 1's should be written in. If the 1's are not written, the pupils may get 0 instead

of 1 when they multiply such fractions as $\frac{2}{3}$ and $\frac{3}{2}$. They may also get 2 instead of $\frac{1}{2}$ when they multiply such fractions as $\frac{2}{3}$ and $\frac{3}{4}$.

After cancelation has been taught, pupils should be urged to use the new method as a means of saving time and effort. However, a pupil should not be penalized heavily if he fails to make use of cancelation on a test. The loss is his loss. Too often, no distinction is made by the teacher between a minor oversight, such as failure to take advantage of an opportunity to cancel, and a major error in the procedure employed or in computation.

THE DIVISION OF FRACTIONS

A difficult topic. Most pupils find division with fractions, particularly division by a fraction, to be a difficult topic. It is difficult for three reasons. First, the procedure is new, so different from anything the pupil has already learned to do that he is baffled and disconcerted by it. Secondly, heretofore in division the quotient has always been smaller than the dividend; now, the pupil discovers to his amazement that if the divisor is a proper fraction the quotient is larger than the dividend, sometimes much larger. Finally, the explanations which some teachers offer (if they offer an explanation) as to why the divisor is inverted are unintelligible to most pupils and serve only to increase the difficulty; furthermore, there may be no clear explanation of how to tell which fraction to invert.

Good teaching will not eliminate these difficulties but it will greatly reduce them. Thorndike says, "If proper treatment is applied, the difficulty is reduced to

perhaps one-quarter or even one-tenth of what it was by the older methods; but even so it demands consideration."[8] With good teaching, this new procedure will seem reasonable and natural, there will be an intelligible explanation of why the divisor is inverted, and the pupil will not be amazed to find quotients larger than dividends.

Pupils' errors in division. The number of errors which Brueckner found in division of fractions was greater than the number which he found in multiplication but not so great as the number which he found in addition or in subtraction. His multiplication and division tests were not given in Grade VA but pupils in Grades VIB and VIA made 4,875 errors in division.[9] Of these, 406 were cases of omitting examples, cases which can hardly be classified as errors.

Brueckner found that 31.1 per cent of the 4,875 errors were instances of performing the wrong operation, 13.8 per cent were errors in computation, 12.1 per cent were errors which indicated a lack of an understanding of the process, 8.9 per cent were instances of failure to reduce results to lowest terms or of errors in reducing, 8.6 per cent represented difficulties in changing mixed numbers to improper fractions, 7.2 per cent were instances of failure to change improper fractions to mixed numbers, 2.3 per cent were errors in copying figures, 1.5 per cent represented difficulties with cancelation, and in 6.0 per cent of the cases the error could not be detected.[10]

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[8] Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, p. 175.

[9] Brueckner, Leo J., *op. cit.*, p. 196.

[10] *Ibid.*, pp. 213-214.

Those pupils who used the wrong process used multiplication; that is, they failed to invert the divisor. A little remedial teaching should eliminate or greatly reduce the frequency of this error. Errors in computation again indicate a need for better teaching of the fundamental operations with integers and for maintenance exercises in these operations. Most of the other errors represent conditions which it should not be very difficult to correct. However, those 590 errors which indicated a lack of an understanding of the process of division with fractions represent a serious condition. Obviously, pupils who invert the dividend instead of the divisor, or invert both the dividend and the divisor, or add numerators or denominators do not understand the process of division with fractions at all. Brueckner's study revealed all of these errors. For them, the subject should be re-taught with emphasis upon meaning at each step.

Types of examples in division. The same types of examples which we enumerated in multiplication may be recognized in division. Again, we may conveniently group these types into three major classes, remembering that in each class the word "fractions" will include both proper fractions and mixed numbers.

1. Fractions divided by integers.
2. Integers divided by fractions.
3. Fractions divided by fractions.

Under (1) we shall have (a) proper fractions divided by integers and (b) mixed numbers divided by integers. Under (2) we shall have (a) integers divided by proper fractions and (b) integers divided by mixed numbers. Under (3) we shall have (a) proper fractions divided

by proper fractions, (b) proper fractions divided by mixed numbers, (c) mixed numbers divided by proper fractions, and (d) mixed numbers divided by mixed numbers. These eight types correspond to the eight types which were stated for multiplication earlier in the chapter. In addition to these, we also have fractions arising from the division of integers by integers.

Division of a fraction by an integer. The teacher may use a problem and develop the subject in some such manner as the following.

TEACHER: "When Arthur, Louise, and Jack came home from school, they were very hungry. 'There is half of an apple pie in the refrigerator,' said their mother. So Arthur cut the half-pie into three equal pieces. What part of a pie did each get. How shall we find out?"

PUPIL: "Divide $\frac{1}{2}$ by 3."

TEACHER: "Yes. What part of the half-pie does each child get?"

PUPIL: "One-third."

TEACHER: "Then we can divide $\frac{1}{2}$ by 3 by finding $\frac{1}{3}$ of $\frac{1}{2}$. So, (writing) $\frac{1}{2} \div 3 = \frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$. We have already learned that 'of' means '$\times$.' So, $\frac{1}{2} \div 3 = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$. Or, we can write, $\frac{1}{2} \div 3 = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ since it does not make any difference whether we write $\frac{1}{3} \times \frac{1}{2}$ or $\frac{1}{2} \times \frac{1}{3}$. Now, let us solve some more examples. What is $\frac{1}{2} \div 4$?"

PUPIL: "$\frac{1}{2} \div 4 = \frac{1}{4}$ of $\frac{1}{2} = \frac{1}{8}$."

Have the pupils solve several similar examples in this fashion. The following and others may be used:

$\frac{1}{2} \div 2 =$	$\frac{1}{3} \div 2 =$	$\frac{1}{4} \div 3 =$
$\frac{1}{2} \div 5 =$	$\frac{1}{3} \div 4 =$	$\frac{1}{4} \div 4 =$
$\frac{1}{2} \div 6 =$	$\frac{1}{3} \div 3 =$	$\frac{1}{4} \div 2 =$

If the point is illustrated concretely or semi-concretely several times and if the teacher goes slowly, most of the pupils will be able to see that dividing by 3 is the same as multiplying by $\frac{1}{3}$, that dividing by 2 is the same as multiplying by $\frac{1}{2}$, etc. Diagrams will help pupils to understand this. A semi-circle may be divided into 2, 3, or 4 equal parts and the pupils may see that each of the parts is $\frac{1}{4}$, $\frac{1}{6}$, or $\frac{1}{8}$ of the entire circle. This kind of illustration will help with the pie problem. Rectangular diagrams may be used for this purpose also.

A few very common fractions with numerators larger than 1 may be used next. Pupils will see that $\frac{3}{4} \div 3$ means $\frac{1}{3}$ of $\frac{3}{4}$, that $\frac{2}{3} \div 2$ means $\frac{1}{2}$ of $\frac{2}{3}$, that $\frac{3}{4} \div 2$ means $\frac{1}{2}$ of $\frac{3}{4}$, etc. Each of these is easily represented by a circular or rectangular diagram. At first, the solution may be written out in this manner,

$$\frac{3}{4} \div 3 = \frac{1}{3} \text{ of } \frac{3}{4} = \frac{1}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{3}}}{4} = \frac{1}{4},$$

but gradually pupils should become accustomed to the other form,

$$\frac{3}{4} \div 3 = \frac{\overset{1}{\cancel{3}}}{4} \times \frac{1}{\underset{1}{\cancel{3}}} = \frac{1}{4},$$

since it represents the easiest and quickest way to solve the example.

It has been suggested that when pupils learn division with integers, they should learn division as *partition* as well as division as *measurement*.[11] That is, they should think of $24 \div 4$ as $\frac{1}{4}$ of 24, etc. Normally, this use of

[11] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades*. New York: Silver Burdett Company, 1937, pp. 282-285.

the fraction form with the division facts comes about two years before division with fractions. This means that the pupils should be thoroughly accustomed to this idea of division by the time they first undertake to divide a fraction by an integer and that they should not find the latter difficult to understand.

When a division example has been changed into a multiplication example, there should be no further difficulty with the solution if the pupil has learned how to multiply fractions. Also, dividing a mixed number by an integer should be no harder than dividing a proper fraction by an integer. A pupil will simply change the mixed number to an improper fraction as has been his practice in solving multiplication examples. He will cancel if he can or reduce his answer to lowest terms and if his answer is an improper fraction, he will change it to a mixed number.

Set 17 illustrates the kinds of examples which may be given for practice. There are 40 examples in this set. Twenty of the 40 examples are cases of dividing a proper fraction by an integer, and 20 have mixed numbers as dividends. Practice is distributed over the fractions of more frequent occurrence with a few of the less common fractions included. The teacher may prepare additional examples similar to these.

Practice Examples in Fractions, Set 17

Division of a Fraction by an Integer

$\frac{1}{2} \div 2 =$	$\frac{1}{4} \div 3 =$	$\frac{1}{3} \div 5 =$	$\frac{1}{5} \div 4 =$
$\frac{1}{8} \div 5 =$	$\frac{1}{6} \div 6 =$	$\frac{1}{7} \div 3 =$	$\frac{1}{12} \div 4 =$
$\frac{2}{3} \div 4 =$	$\frac{2}{5} \div 7 =$	$\frac{5}{8} \div 3 =$	$\frac{4}{5} \div 2 =$

$\frac{3}{8} \div 6 =$	$\frac{5}{6} \div 4 =$	$\frac{3}{5} \div 2 =$	$\frac{7}{8} \div 4 =$
$\frac{3}{4} \div 2 =$	$\frac{2}{7} \div 3 =$	$\frac{5}{12} \div 4 =$	$\frac{3}{10} \div 2 =$
$4\frac{1}{2} \div 3 =$	$6\frac{1}{4} \div 2 =$	$3\frac{3}{8} \div 3 =$	$2\frac{4}{5} \div 5 =$
$6\frac{7}{8} \div 4 =$	$3\frac{1}{7} \div 2 =$	$4\frac{1}{3} \div 5 =$	$5\frac{1}{6} \div 3 =$
$2\frac{3}{10} \div 5 =$	$6\frac{1}{8} \div 3 =$	$7\frac{3}{4} \div 4 =$	$9\frac{5}{8} \div 10 =$
$2\frac{1}{5} \div 3 =$	$4\frac{2}{3} \div 6 =$	$4\frac{7}{12} \div 5 =$	$2\frac{2}{5} \div 2 =$
$8\frac{5}{6} \div 7 =$	$12\frac{3}{5} \div 5 =$	$6\frac{2}{9} \div 4 =$	$5\frac{5}{12} \div 4 =$

Division of an integer by a fraction. The development of the division of an integer by a fraction should indicate the similarity of this kind of division to the division of an integer by an integer. It has been suggested that the pupil should think of such an expression as $4\overline{)12}$ as asking the question, "How many 4's in 12?"[12] If this form of expression has not been stressed in early work in division, it should be stressed now. This is the measurement idea of division. Then, when pupils have become thoroughly accustomed to thinking of $4\overline{)12}$, $2\overline{)8}$, etc., (or $12 \div 4 =$ $8 \div 2 =$), as asking "How many 4's in 12?, How many 2's in 8?" etc., they may read such an example as $3 \div \frac{1}{2} =$ as asking, "How many halves in 3?"

To illustrate further this point of view as to the meaning of division, such a problem as the following may be used.

In making up Christmas stockings for poor children, the Salvation Army is putting 2 oranges in each stocking. How many stockings can be supplied with 6 dozen (72) oranges?

As the problem is solved, it should be emphasized that we find how many stockings can be supplied by finding

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[12] Morton, R. L., *op. cit.*, p. 217.

how many 2's there are in 72. Then, a problem involving the division of an integer by a fraction, such as the following, may be used.

If each person is served one-half of an orange for breakfast, how many persons can a restaurant serve with two dozen (24) oranges?

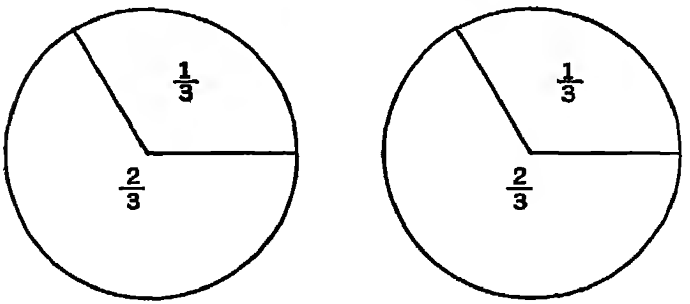
And as this problem is solved, it should be emphasized that we find how many persons can be served by finding how many halves there are in 24. This, again, is the measurement idea of division.

Probably the greatest single aid toward the development of an understanding of the meaning of dividing an integer by a fraction is this interpretation of the meaning of a division example. Too often, pupils who see such an example as $3 \div \frac{1}{4}$ see nothing but two numbers and an indication that one is to be divided by the other. But if $3 \div \frac{1}{4}$ is read as, "How many fourths in 3?" it takes on meaning and the answer, 12, ceases to be a mystery.

Diagrams should be used freely to illustrate the division of an integer by a fraction. Pies can be drawn and cut into halves, thirds, fourths, fifths, sixths, and eighths. Rectangles of varying lengths can be carefully drawn and divided into half-inch pieces, one-fourth inch pieces, etc. This semi-concrete work, combined with the use of meaningful problems, will help the pupil to understand what actually is done when an integer is divided by a fraction. If the numerator of the fraction is 1, he will soon see that he multiplies the dividend (the integer) by the denominator of the divisor to find the number of parts of the given size in the number of given wholes.

After a number of examples in which integers are divided by unit fractions are solved, examples requiring the division of integers by other proper fractions may be studied. The first of these should be examples yielding integral quotients for this kind is more easily made meaningful by diagrams. We may begin with the example, $2 \div 2/3 =$. Draw on the blackboard two circles of equal size and mark off two-thirds of each. To answer the question, "How many 2/3's in 2," the pupil observes that there is a 2/3 in the first circle, a 2/3 in the second circle, and that another 2/3 can be made by combining the 1/3 left in the first circle with the 1/3 left in the second circle. Thus, he discovers that $2 \div 2/3 = 3$. Other similar examples may be illustrated and solved in the same way.

FIGURE 7. THE DIVISION OF 2 BY 2/3.



The division of an integer by a fraction can also be made meaningful by changing the integer to a fraction having a denominator equal to that of the divisor. Thus, to divide 2 by 2/3, the pupil changes 2 to thirds and writes $2 \div 2/3 = 6/3 \div 2/3 = 6$ thirds $\div$ 2 thirds $= 3$.

Now, it may be pointed out that there is a short way to get the same answer, that an integer may be divided by a fraction by multiplying the integer by the fraction inverted. The pupil learns that to invert $\frac{2}{3}$ means to write it as $\frac{3}{2}$, that to invert $\frac{1}{4}$ means to write it as $\frac{4}{1}$ or 4, etc. This short method is then applied to the examples which have been solved by diagrams or by changing them to common denominator form and confidence in the short method arises from the observation that in every case the short method produces a result equal to that which has already been secured.

Having learned that to invert $\frac{1}{4}$ means to write it as $\frac{4}{1}$ or 4, the pupil will readily see that to invert 4 or $\frac{4}{1}$ means to write it as $\frac{1}{4}$. Then we may carry the short method which has just been learned back to the examples in which fractions were divided by integers. Back there, the pupils learned that $\frac{2}{3} \div 3 = \frac{2}{3} \times \frac{1}{3}$. Now, they see that they were inverting the divisor when they solved such examples. Thus, the generalization that *to divide by a whole number or by a fraction, one can invert the divisor and multiply*, is strengthened.

We have observed that one of the reasons for the difficulty of division with fractions is the fact that pupils can not understand how the quotient can be larger than the dividend. Even students in teacher-training classes, when confronted with $16 \div \frac{1}{4}$, often write 4 instead of 64 as the quotient. Thorndike senses the pupil's difficulty when he says, "Put yourself in the place of a child who has divided thousands of times and on every occasion found the answer to be much smaller than the number divided—who has had 'make much smaller' as the one uniform associate of 'divide.' He now

is told so to operate that $16 \div \frac{1}{8}$ gives a result far greater than 16, that $1\frac{2}{3} \div \frac{3}{8}$ gives a result much greater than $1\frac{2}{3}$. There is a natural and, in a sense, a commendable reluctance to attach confidence to a procedure that produces a result so contrary to what he has always found previously to be the essence of division. This lack of confidence is very unfavorable to learning."[18]

The development which has been suggested in the preceding pages should enable the child to understand why the quotient may be larger than the dividend. Rather than write 4 as the quotient of 16 divided by $\frac{1}{4}$ because the answer, 4, seems to accord with his earlier experiences in division or learn by rote and follow blindly the rule, "Invert and multiply," the pupil should understand that 64 is a reasonable answer for this example because the example asks him to find out how many fourths there are in 16.

The fact that when the divisor is less than 1, the quotient is larger than the dividend should be discovered by the pupils from their experiences with division. After the fact is discovered, it should be emphasized occasionally and should be associated with two other observations: (1) when the divisor is 1, the quotient is equal to the dividend; and (2) when the divisor is more than 1, the quotient is smaller than the dividend.

Suggestions for practice examples in the division of an integer by a proper fraction or by a mixed number are given in Set 18. In the first 20 of the 40 examples of this set, the divisor is a proper fraction; in the remaining 20, it is a mixed number. Most of the fractions are fractions

[18] Thorndike, Edward Lee, *op. cit.*, pp. 152-153.

of common occurrence but again sufficient variety has been introduced to let the pupil see the general application of the method employed and to enable him to cope with any proper fraction or mixed number which he is likely to encounter in examples of this kind. As in the examples of Set 17, the pupils will convert the mixed numbers to improper fractions, of course.

Practice Examples in Fractions. Set 18

Division of an Integer by a Fraction

$3 \div \frac{1}{2} =$	$2 \div \frac{1}{3} =$	$5 \div \frac{1}{4} =$	$4 \div \frac{1}{5} =$
$4 \div \frac{1}{8} =$	$5 \div \frac{2}{3} =$	$3 \div \frac{1}{6} =$	$6 \div \frac{5}{8} =$
$3 \div \frac{1}{12} =$	$8 \div \frac{1}{7} =$	$2 \div \frac{4}{5} =$	$5 \div \frac{2}{5} =$
$6 \div \frac{7}{8} =$	$9 \div \frac{5}{6} =$	$1 \div \frac{2}{7} =$	$5 \div \frac{3}{10} =$
$3 \div \frac{3}{8} =$	$5 \div \frac{3}{4} =$	$4 \div \frac{5}{12} =$	$7 \div \frac{3}{5} =$
$4 \div 2\frac{1}{2} =$	$5 \div 1\frac{7}{8} =$	$4 \div 3\frac{1}{7} =$	$1 \div 4\frac{1}{4} =$
$7 \div 3\frac{1}{3} =$	$6 \div 5\frac{4}{5} =$	$9 \div 1\frac{3}{8} =$	$10 \div 2\frac{1}{6} =$
$6 \div 3\frac{5}{8} =$	$2 \div 4\frac{1}{8} =$	$6 \div 2\frac{3}{4} =$	$9 \div 5\frac{3}{10} =$
$8 \div 3\frac{7}{12} =$	$1 \div 4\frac{2}{3} =$	$3 \div 1\frac{1}{5} =$	$5 \div 2\frac{2}{5} =$
$6 \div 4\frac{5}{6} =$	$5 \div 2\frac{4}{9} =$	$6 \div 8\frac{3}{5} =$	$6 \div 1\frac{7}{12} =$

Division of a fraction by a fraction. An effort has been made to outline a plan for teaching pupils to divide fractions by integers and integers by fractions which would be meaningful to them. The same or a similar method of rationalization can be extended to the case of division of a fraction by a fraction. This is most easily done with familiar fractions which yield whole numbers for quotients. Thus, pupils can see that $\frac{1}{2} \div \frac{1}{4} = 2$ by thinking, "How many fourths (or quarters) in one-half?" They will have more difficulty in seeing that $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$ even when diagrams are

used to represent the division for pupils, when asked how many halves there are in one-fourth are likely to reply to the effect that there are not any halves in one-fourth. Some may see that there is one-half of one-half in one-fourth but there is likely to be a confusion with multiplication and a lack of clear comprehension. It will be still more difficult for them to see that $\frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$ or that $\frac{1}{3} \div \frac{1}{4} = 1\frac{1}{3}$ although the brightest may even see this when it is represented diagrammatically.

The duller pupils probably will not understand very well any explanation of why the divisor is inverted. It may be that the best which the teacher can do for such pupils is illustrate the proper procedure and allow them to learn it by imitation. However, it is a fair question as to whether pupils who can not understand an arithmetic process should undertake to acquire skill in the process at all. This may be a part of arithmetic which dull pupils should not undertake.

The inability to get dull pupils to understand an explanation of the procedure employed in division with fractions has led many teachers to give up all efforts at rationalization and simply to dictate to all of the pupils the steps which they should follow. It should be remembered, however, that skill with understanding is much more educative than skill alone and that those pupils who can follow the explanations given in the preceding pages will know the processes with fractions better than if they follow blindly adult-made rules for these processes. Furthermore, pupils of average ability, as well as those of superior ability, grasp these explanations well enough to make the procedure reasonable.

Consequently, there is less likelihood of silly mistakes and ridiculous blunders.

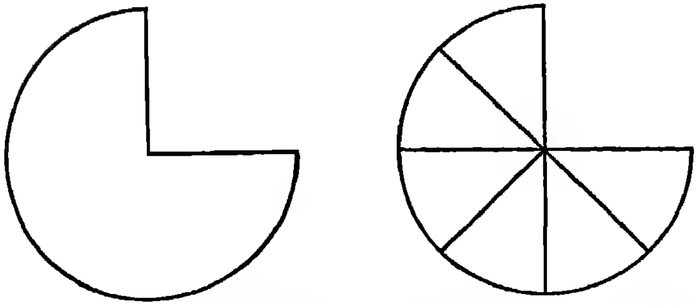
Since the pupils have already learned to divide a fraction by an integer or an integer by a fraction by inverting the divisor and multiplying, they are not taking very much for granted if they generalize to the extent that the same procedure applies to the division of a fraction by a fraction. If this generalization is seen to be sound for a few simple examples, such as $2/3 \div 1/3 = 2$, $3/4 \div 1/4 = 3$, and $1/2 \div 1/4 = 2$, it can very properly be assumed to apply to any example in which a fraction is to be divided by a fraction. Of course, this will apply equally well to the division of a proper fraction by a mixed number, a mixed number by a proper fraction, and a mixed number by a mixed number, since the mixed numbers will all be changed to improper fractions.

However, it is well to use diagrams occasionally to represent divisions which lend themselves readily to such representation. Let us suppose that the pupils are solving this problem:

Mother baked a pie and cut one-fourth out for dad's lunch. Then, she cut the rest of the pie into eighths. That is, one piece was one-eighth of the whole pie. How many pieces were left?

A drawing on the blackboard, like the diagram at the left in Figure 8, helps pupils to visualize the appearance of the pie after one-fourth had been removed. They will see that three-fourths remain. Then lines are drawn to show the three fourths and more lines are drawn to show the remainder of the pie cut into eighths. The figure now appears as at the right in Figure 8. The an-

FIGURE 8. THE DIVISION OF $\frac{3}{4}$ BY $\frac{1}{8}$.



swer to the problem is seen to be 6. Now the example is solved by inverting the divisor and multiplying,

$$\frac{3}{4} \div \frac{1}{8} = \frac{3}{\cancel{4}_{1}} \times \frac{\cancel{8}^{2}}{1} = 6.$$

When either an integer or a fraction has been divided by a fraction, the pupil may assure himself as to the accuracy of his work by employing multiplication as a check. He has learned to use this check when dividing with integers. The following examples will illustrate the use of this check.

Example	Check
$\begin{array}{r} 65 \\ 7\overline{)455} \\ 42 \\ \hline 35 \\ 35 \\ \hline \end{array}$	$\begin{array}{r} 65 \\ 7 \\ \hline 455 \end{array}$
$3 \div \frac{3}{4} = \cancel{3}^{1} \times \frac{4}{\cancel{3}_{1}} = 4$	$\frac{\cancel{4}^{1}}{1} \times \frac{3}{\cancel{4}_{1}} = 3$

$2\frac{1}{2} \div 1\frac{1}{4} = \frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{2}}} \times \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{5}}} = 2 \qquad 2 \times 1\frac{1}{4} = 2\frac{1}{2}$

$3\frac{1}{2} \div 1\frac{1}{3} = \frac{7}{2} \times \frac{3}{4} = \frac{21}{8} = 2\frac{5}{8} \qquad 2\frac{5}{8} \times 1\frac{1}{3} =$

$\frac{\overset{7}{\cancel{21}}}{\underset{2}{\cancel{8}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{3}}} = \frac{7}{2} = 3\frac{1}{2}$

Typical examples for practice are shown in Sets 19 and 20. Each of these sets contains 60 examples. In Set 19, we have the division of a proper fraction

Practice Examples in Fractions. Set 19

Division of a Fraction by a Fraction

$\frac12 \div \frac14 =$	$\frac12 \div \frac18 =$	$\frac13 \div \frac16 =$	$\frac14 \div \frac18 =$
$\frac12 \div \frac13 =$	$\frac12 \div \frac34 =$	$\frac23 \div \frac13 =$	$\frac14 \div \frac12 =$
$\frac34 \div \frac12 =$	$\frac34 \div \frac38 =$	$\frac23 \div \frac16 =$	$\frac13 \div \frac14 =$
$\frac56 \div \frac34 =$	$\frac12 \div \frac25 =$	$\frac14 \div \frac16 =$	$\frac23 \div \frac38 =$
$\frac13 \div \frac18 =$	$\frac12 \div \frac12 =$	$\frac58 \div \frac56 =$	$\frac34 \div \frac23 =$
$\frac13 \div \frac12 =$	$\frac34 \div \frac35 =$	$\frac23 \div \frac34 =$	$\frac14 \div \frac34 =$
$\frac12 \div \frac58 =$	$\frac13 \div \frac34 =$	$\frac58 \div \frac23 =$	$\frac13 \div \frac78 =$
$\frac16 \div \frac18 =$	$\frac34 \div \frac58 =$	$\frac23 \div \frac12 =$	$\frac14 \div \frac14 =$
$\frac12 \div \frac16 =$	$\frac14 \div \frac45 =$	$\frac34 \div \frac56 =$	$\frac14 \div \frac35 =$
$\frac13 \div 2\frac25 =$	$\frac38 \div 3\frac13 =$	$\frac14 \div 1\frac23 =$	$\frac23 \div 1\frac35 =$
$\frac14 \div 2\frac56 =$	$\frac12 \div 4\frac38 =$	$\frac23 \div 1\frac78 =$	$\frac56 \div 2\frac12 =$
$\frac12 \div 3\frac17 =$	$\frac23 \div 1\frac18 =$	$\frac14 \div 6\frac13 =$	$\frac13 \div 3\frac13 =$
$\frac14 \div 6\frac78 =$	$\frac14 \div 3\frac15 =$	$\frac13 \div 7\frac45 =$	$\frac23 \div 4\frac56 =$
$\frac34 \div 5\frac13 =$	$\frac23 \div 4\frac14 =$	$\frac12 \div 3\frac56 =$	$\frac13 \div 5\frac38 =$
$\frac78 \div 3\frac19 =$	$\frac34 \div 2\frac45 =$	$\frac13 \div 6\frac58 =$	$\frac34 \div 7\frac15 =$

by a proper fraction and of a proper fraction by a mixed number—36 examples of the former and 24 examples of the latter. In Set 20, we have provided for the di-

vision of a mixed number by a proper fraction and for the division of a mixed number by a mixed number—32 examples of the former and 28 examples of the latter.

PRACTICE EXAMPLES IN FRACTIONS. SET 20

Division of a Fraction by a Fraction

$2\frac{1}{2} \div \frac{2}{3} =$	$5\frac{1}{3} \div \frac{2}{3} =$	$9\frac{3}{4} \div 3\frac{1}{4} =$
$4\frac{3}{4} \div \frac{1}{8} =$	$4\frac{1}{4} \div \frac{3}{8} =$	$6\frac{1}{2} \div 5\frac{1}{3} =$
$7\frac{1}{3} \div \frac{3}{5} =$	$2\frac{7}{8} \div \frac{5}{9} =$	$1\frac{3}{4} \div 2\frac{2}{3} =$
$9\frac{3}{4} \div \frac{7}{8} =$	$1\frac{5}{12} \div \frac{2}{3} =$	$6\frac{1}{4} \div 3\frac{3}{5} =$
$6\frac{1}{4} \div \frac{3}{7} =$	$5\frac{2}{3} \div \frac{1}{5} =$	$6\frac{3}{4} \div 9\frac{3}{5} =$
$5\frac{2}{3} \div \frac{2}{5} =$	$3\frac{1}{6} \div \frac{1}{4} =$	$3\frac{1}{4} \div 1\frac{3}{4} =$
$4\frac{1}{6} \div \frac{1}{2} =$	$5\frac{1}{4} \div \frac{4}{5} =$	$5\frac{2}{3} \div 1\frac{3}{4} =$
$4\frac{1}{8} \div \frac{1}{3} =$	$3\frac{1}{4} \div \frac{5}{8} =$	$3\frac{1}{3} \div 6\frac{1}{8} =$
$8\frac{1}{2} \div \frac{4}{5} =$	$8\frac{3}{4} \div \frac{1}{4} =$	$8\frac{1}{4} \div 4\frac{1}{8} =$
$1\frac{3}{4} \div \frac{1}{6} =$	$10\frac{2}{3} \div \frac{5}{8} =$	$5\frac{1}{2} \div 2\frac{1}{6} =$
$2\frac{1}{6} \div \frac{1}{8} =$	$7\frac{3}{4} \div \frac{2}{5} =$	$2\frac{2}{3} \div 1\frac{3}{8} =$
$3\frac{3}{4} \div \frac{3}{4} =$	$6\frac{3}{4} \div \frac{7}{12} =$	$1\frac{2}{7} \div 1\frac{1}{3} =$
$2\frac{3}{8} \div \frac{1}{2} =$	$3\frac{1}{2} \div 4\frac{1}{2} =$	$4\frac{5}{8} \div 2\frac{2}{3} =$
$4\frac{2}{3} \div \frac{2}{3} =$	$6\frac{1}{4} \div 4\frac{1}{2} =$	$4\frac{1}{8} \div 1\frac{1}{4} =$
$6\frac{1}{3} \div \frac{1}{5} =$	$2\frac{3}{4} \div 4\frac{1}{2} =$	$1\frac{2}{3} \div 3\frac{1}{2} =$
$5\frac{1}{3} \div \frac{5}{6} =$	$7\frac{1}{4} \div 3\frac{1}{6} =$	$2\frac{2}{3} \div 1\frac{1}{3} =$
$6\frac{1}{2} \div \frac{1}{5} =$	$4\frac{5}{6} \div 3\frac{1}{3} =$	$9\frac{1}{4} \div 3\frac{1}{4} =$
$2\frac{1}{4} \div \frac{2}{5} =$	$12\frac{1}{2} \div 3\frac{1}{4} =$	$5\frac{1}{3} \div 2\frac{1}{2} =$
$4\frac{1}{2} \div \frac{3}{5} =$	$4\frac{1}{3} \div 2\frac{1}{3} =$	$9\frac{1}{4} \div 6\frac{7}{8} =$
$7\frac{1}{2} \div \frac{7}{8} =$	$7\frac{1}{3} \div 2\frac{1}{4} =$	$7\frac{1}{2} \div 1\frac{3}{4} =$

The entire 120 examples provide for a rather wide variety of combinations of fractions. Used in conjunction with Sets 17 and 18 they should give the pupil an amount and variety of practice sufficient to enable

him to solve any examples in division with fractions which he may encounter.

Division of an integer by an integer. The division of integers has been discussed in Chapter 3. It is recognized that a remainder sometimes should be left as a remainder and that sometimes it should be expressed as a fractional part of the quotient making the quotient a mixed number. The manner in which remainders are disposed of should be determined by the nature of the problems from which the division examples are derived.

When studying fractions, pupils should see that one meaning of a fraction is that of an indicated division. Thus, $3/5$ means $3 \div 5$, $5/3$ means $5 \div 3$, etc. There should be practice in interpreting fractions in this manner. This practice should include fractions which require reducing to lowest terms and changing to mixed numbers.

It should be noted that if a remainder in division has been expressed as a fractional part of the quotient, the pupil, in a way, changes a mixed number to an improper fraction when he checks the example. Thus, if he has divided 19 by 5 and has obtained the quotient, $3\frac{4}{5}$, he multiplies 3 by 5 and adds 4 to the product when he checks the example. All that he needs to do now is to write 5 under the 19 to accomplish in full the transformation of a mixed number into an improper fraction. Then, when he sees that he can indicate the division of 19 by 5 by writing the fraction, $19/5$, the circuit is complete.

To attain adequate skill in the fundamental operations with fractions requires practice—more practice than some pupils get. In the 20 sets of practice examples

given in Chapters 4, 5, and 6, there are, in all, 837 examples. These 837 examples may be more than are needed by some pupils, they may constitute an optimum amount for others, and they may be quite insufficient for those who learn fractions with great difficulty. The responsibility for determining the amount of practice which the individual pupils need must rest upon the teacher for it will vary greatly from pupil to pupil in a class. Grouping the pupils according to their ability and their skill will make it much easier to adapt instruction to the requirements of the individual members of the group.

Ordinarily, if a skill is worth acquiring, it is worth acquiring for permanent retention. But pupils sometimes forget the operations with fractions after having learned them. It is not unusual to find pupils in one grade unable to solve examples in fractions as well as do the pupils in the preceding grade in the same school. The fault sometimes lies in the fact that we have failed to provide properly for continued practice; we have failed to maintain these abilities at a desirable level after having helped the pupils to acquire them. Pupils should not only receive a sufficient *amount* of practice, then, but this practice should be *properly distributed*. Practice should continue through the years of the junior high school. Good teaching requires a carefully planned maintenance program.

QUESTIONS AND REVIEW EXERCISES

1. How do you account for the fact that pupils frequently make better scores on tests in multiplication with fractions than on tests in the other operations? Can one

say that they understand multiplication better than they understand addition or subtraction?

2. Just what is the difference between the printed signs for *plus* and *times?* Why should children confuse these signs?

3. Would you have multiplication with fractions taught before addition or subtraction? Why?

4. How do you account for the fact that pupils whose papers Brueckner examined made fewer errors in multiplication than in addition, subtraction, or division?

5. What were the commonest errors which Brueckner found in multiplication with fractions? Which of these groups of errors is the most serious?

6. Is it to be expected that if the papers of another 200 pupils were examined for the errors which Brueckner found, they would be found to occur again with about the same frequency? Why?

7. Can a satisfactory analysis of pupils' errors be made from test papers? Why?

8. State the eight groups of examples suggested for multiplication. How are these eight groups classified again into three larger groups?

9. Why is multiplication with fractions begun with the multiplication of a fraction by an integer? Do you believe that it would have been as good a plan to begin with the multiplication of an integer by a fraction?

10. How would you help pupils to understand that when a fraction is to be multiplied by an integer, only the numerator of the fraction is to be multiplied by the integer?

11. What is the danger in teaching cancelation at the very beginning of the pupil's work in multiplication with fractions?

12. In multiplying a mixed number by an integer, should the equation form or the column form be taught first?

Which is the better form for final permanent use?

13. In solving an example requiring the multiplication of an integer by a mixed number, does it matter if the pupil multiplies the mixed number by the integer? Why?

14. What advantage is there in making considerable use of the "of" form of statement in early work in the multiplication of an integer by a fraction?

15. Ordinarily, can you say that telling is teaching? Is telling ever teaching?

16. Describe an introductory lesson which you would teach in the multiplication of a fraction by a fraction.

17. Were you taught to change mixed numbers to improper fractions by simply following a rule which you learned by rote? How would you teach this topic?

18. Would you ever require pupils to multiply mixed numbers by mixed numbers by using the column form of statement rather than the equation form?

19. Mathematically, what takes place in cancelation? If one cancels all he can, does he ever have to reduce his products to lower terms? If $\frac{1}{2}$ is multiplied by $\frac{2}{3}$ why should 1's be written when the 2's are canceled?

20. What reasons are given for the difficulty of division with fractions?

21. How can good teaching make a topic easier if it can not make it easy?

22. What was the outstanding error found by Brueckner in division with fractions? How do you account for the prevalence of this error?

23. Describe an introductory lesson which you would use in teaching pupils to divide a fraction by an integer.

24. Some teachers begin teaching the division of a fraction by an integer by writing the examples like this, $\frac{1}{2} \div \frac{3}{1} =$. Then they instruct the pupils to change the division sign to a multiplication sign and to invert the divisor. Criticize this procedure.

25. How can work in the division of fractions by integers be tied up with the division facts which were learned probably two years earlier?

26. What is meant by division as *partition?* By division as *measurement?* Are both of these ideas of division involved in division with fractions?

27. How would you help a pupil to understand that the quotient of $8 \div \frac{1}{2}$ is 16 and not 4? What makes this type of examples so difficult for pupils?

28. Can a foot rule be used to illustrate the multiplication of such fractions as $\frac{1}{2}$ and $\frac{1}{2}$, or $\frac{1}{3}$ and $\frac{1}{2}$? Can a foot rule be used to illustrate division with fractions?

29. How can a pupil be helped to understand that if the divisor is less than 1 the quotient is larger than the dividend although most of his division examples have yielded quotients which were smaller than the dividends?

30. To what extent would you use diagrams and other concrete or semi-concrete aids to help pupils see why they should invert the divisor and multiply when dividing a fraction by a fraction?

31. If pupils are too dull to understand the process of division with fractions, should they be excused from the study of this subject?

32. What is meant by the statement that a fraction is an indicated division? When should the remainder in a division example be expressed as a fractional part of the quotient?

33. How may checking a division example lead to the changing of a mixed number to an improper fraction?

34. Do you believe that if a skill is worth acquiring, it is worth acquiring for permanent retention? Is this always true?

35. How should practice be distributed in a correctly planned maintenance program?

CHAPTER TEST

Read each statement and decide whether it is true or false. A key for this test will be found on pages 533-534.

1. A pupil who understands nothing of the operations with fractions usually will make a higher score on a test in multiplication than on a test in any one of the other three operations.

2. Division is the hardest of the operations with fractions.

3. The author recommends that multiplication of fractions should be taught before addition or subtraction.

4. Brueckner found fewer errors in multiplication than in either addition or subtraction.

5. An error in computation is more serious than an error indicating a failure to understand the process.

6. The per cent of errors of a given type in one school is likely to be almost the same as the per cent of errors of the same type in another school.

7. In the example, $4 \times 2/3 =$, an integer is to be multiplied by a fraction.

8. An ideal analysis of errors can be made from pupils' test papers.

9. Cancelation should be introduced early in the pupil's work in multiplication with fractions.

10. When a pupil is taught to multiply $2/3$ by 3, he should be taught to write the 3 as $3/1$.

11. In multiplying $4\frac{1}{5}$ by 3, it was recommended that the pupil first multiply $1/5$ by 3.

12. Arithmetically, multiplying an integer by a fraction is almost the same as multiplying a fraction by an integer.

13. In the first examples requiring that an integer be multiplied by a fraction, the fraction should be a proper fraction.

14. The routine of multiplying a fraction by a fraction is easily learned.

15. Ordinarily, telling is not teaching.

16. It is possible to show by a rectangular diagram that $\frac{3}{4}$ of $\frac{1}{3}$ is $\frac{1}{4}$.

17. In multiplying a fraction by a mixed number, the pupils should change the mixed number to an improper fraction.

18. In teaching pupils to change a mixed number to an improper fraction, the teacher should first state the rule: "Multiply the integer by the denominator of the fraction, add the numerator to this product, and write the denominator under the sum."

19. It is impossible to multiply a mixed number by a mixed number without changing both mixed numbers to improper fractions.

20. If one cancels all he can, the product never has to be reduced to lower terms.

21. If one cancels all he can, the product never has to be changed to a mixed number.

22. In cancelation, one reduces to lower terms before multiplying instead of after multiplying.

23. In a division example, the quotient is always smaller than the dividend.

24. Good teaching will make division easy.

25. The commonest error which Brueckner found in division with fractions was performing the wrong operation.

26. All of the types of examples recognized in multiplication of fractions were recognized also in division.

27. Division as measurement requires the finding of a part of a number.

28. In teaching the division of an integer by a fraction, emphasis should be placed upon the measurement idea of division.

29. One can always divide by a fraction by multiplying by the reciprocal of the fraction.

30. The rule, "Invert and multiply," should be given early in the pupil's work in division with fractions.

31. In learning division with fractions, pupils should make generalizations as to the procedure involved.

32. Diagrams may be used to show that $1\frac{1}{2}$ divided by $\frac{1}{2}$ equals 3.

33. In division with integers, the remainder should always be expressed as a fractional part of the quotient.

34. Ordinarily, if a skill is worth acquiring, it is worth acquiring for permanent retention.

35. It should not be necessary to provide practice on fractions in the junior high school.

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10. Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937, 591 pp. Multiplication and division with fractions are treated briefly on pages 395-402.

11. Wilderman, Edward. *The Teaching of Fractions.* Chicago: The Plymouth Press, 1923, 145 pp. Chapter IV, entitled "Fractions in Multiplication and Division," pages 78-114, presents various phases of this subject.

CHAPTER 7

DECIMAL FRACTIONS

The approach to decimal fractions. It seems best to approach the study of decimal fractions through the medium of related common fractions. Decimal fractions are simply common fractions whose denominators are 10, 100, 1000, or some integral power of 10 greater than 0.

In order that this approach may be used effectively, pupils must have a well developed understanding of the meaning of common fractions before decimal fractions are introduced. It is not uncommon to find pupils with a pitifully meager grasp of common fractions trying to understand the meaning of decimals. To learn by rote in one phase of arithmetic often means that pupils must learn by rote in later related phases or not learn at all. The effects of rote learning are disastrously cumulative. The effects of rational learning are also cumulative but they lead to broader and richer understandings and to wider horizons.

If decimal fractions are introduced as a special case of common fractions, the subject may be developed in the following manner.

TEACHER: "Read this fraction, $3/10$."

PUPIL: "Three tenths."

TEACHER: "We have another way of writing fractions whose denominator is 10. We simply leave off the denominator and place a period, called a point, in front of the

numerator. The fraction, three-tenths, can then be written like this, .3. So,

$$^{3}/_{10} = .3$$

And,

$$^{4}/_{10} = .4$$
$$^{7}/_{10} = .7$$
$$^{9}/_{10} = .9, \text{ etc.}$$

We read .3 just as we read $^{3}/_{10}$. Each is read, "three-tenths." Now, read this fraction, $^{25}/_{100}$."

PUPIL: "Twenty-five hundredths."

TEACHER: "If the denominator of a fraction is 100, we can drop the denominator and place a point before the numerator. Then,

$$^{25}/_{100} = .25$$

And,

$$^{62}/_{100} = .62$$
$$^{43}/_{100} = .43$$

We read .25 just as we read $^{25}/_{100}$. Each is read, "twenty-five hundredths."

If the denominator is 10 or 100, then, we can drop the denominator and place a point in the numerator. Fractions like .4, .2, .25, .87, etc., are called *decimal fractions,* or simply *decimals.*

In the next lesson the teacher will emphasize the fact that if the fraction is tenths there is just one figure at the right of the decimal point but that if it is hundredths there are always two figures at the right of the decimal point. Then fractions like $^{5}/_{100}$ will be introduced and the necessity of a zero after the decimal point as a means of distinguishing $^{5}/_{100}$ from $^{5}/_{10}$ in decimal form will be emphasized. Fractions of three or more decimal places will then be taught.

It is desirable that the first work with decimal frac-

tions as such be "touched off" by some observation or experience in which decimal fractions are found. Mileages in railway time tables are usually given to one or two places of decimals. Distances on highway signs are sometimes recorded to one decimal place. Automobile odometers usually record mileage to one decimal place. Boys hear conversation about 15 thousandths of an inch of clearance in adjusting valve stems, about fitting an old car with new pistons which are seven thousandths of an inch oversize, etc.

Another approach to decimal fractions. Decimals may also be taught as an extension of our system of notation to tenths, hundredths, thousandths, etc. The success of this approach depends upon how well the pupils understand the number system. They should have learned by this time that the number system is a decimal system although they may not have learned the word "decimal." That is, they should know and understand that such a number as 3333 means $3000 + 300 + 30 + 3$ and that the 3 in hundreds' place, for example, has a value ten times as great as the value of the 3 at the right and one-tenth as great as the value of the 3 at the left. But it was stated in Chapter 2 that many pupils do not readily grasp the significance of figures at the right of the decimal point simply because they do not understand the significance of figures at the left of the decimal point. Pupils can hardly be expected to understand decimal fractions if they do not understand decimal whole numbers.

In teaching decimal fractions as an extension of our system of notation to tenths, hundredths, thousandths, etc., the lesson may be developed as follows:

TEACHER: "Read this number, 324."

PUPIL: "Three hundred twenty-four."

TEACHER: "What do we call the place which the 4 occupies?"

PUPIL: "Units' place."

TEACHER: "And the place the 2 occupies?"

PUPIL: "Tens' place."

TEACHER: "And the 3?"

PUPIL: "Hundreds' place."

TEACHER: "Now we can place a period, called a point, after the 4 and write more figures at the right of the point. Let us write, 324.65. The first figure at the right of the 4 stands for *tenths* and the second figure at the right of the 4 stands for *hundredths*. Then we have 6 tenths and 5 hundredths, or 65 hundredths. So we read the number, 324.65, like this: *three hundred twenty-four and sixty-five hundredths*. The sixty-five hundredths is a fraction and, when written like this, is called a decimal fraction. We can also write it like this, $324\frac{65}{100}$. So, $324.65 = 324\frac{65}{100}$. How do we read this number if it has a dollar sign in front of it, like this, $324.65?"

PUPIL: "Three hundred twenty-four dollars and sixty-five cents."

TEACHER: "This will help you to see what 324.65 means. When we put the dollar sign on, the 65 means cents. But there are how many cents in a dollar?"

PUPIL: "One hundred."

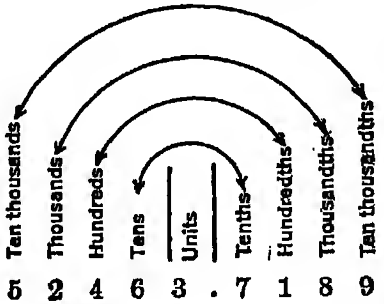
TEACHER: "Then 65 cents is what part of a dollar?"

PUPIL: "Sixty-five hundredths."

The teacher will then emphasize the fact that when there are two figures at the right of the point, called the decimal point, the fraction is always hundredths. And, by carrying the analogy to money further, the teacher will show that the 6 means 6 dimes and that the

5 means 5 cents. But since there are 10 dimes in a dollar and 100 cents in a dollar, the first figure at the right of the decimal point always means tenths and the second, hundredths. To show how a number like 42.07 is written, this amount will be written as money with the familiar dollar sign and the necessity for a 0 after the decimal point in such cases will be stressed.

FIGURE 9. PLACE SIGNIFICANCE IN DECIMAL MIXED NUMBERS



After practice in reading and writing decimal mixed numbers and decimal fractions of one and two decimal places has been provided, the work will be extended to include thousandths, ten thousandths, etc. The analogy of the number words used for the figures at the right of the point to those used at the left may be emphasized in the manner shown by Figure 9. In teaching children to read decimals by this method, we make the units' place, rather than the decimal point, the point of reference. The analogy of tens to tenths, of hundreds to hundredths, etc., as we go to the left and to the right of units' place one step, two steps, etc., is thus made clear. This is emphasized in Figure 9 by drawing vertical lines at the left and right of the 3 in units' place.

Which is the better approach? There seems to be no experimental evidence as to which of these two approaches to the study of decimal fractions is the better. It seems clear to the author that both approaches should be used and that one should supplement the other in developing an understanding of decimal fractions. Because the arithmetic course is usually organized so that decimal fractions are studied soon after common fractions, it seems best to begin by noting the relationship of decimal fractions to common fractions and to follow this after a time with the other approach to the study of decimals.

The study of decimal fractions and decimal mixed numbers should broaden the pupil's understanding of the number system in general. He should see that in any such number the value of any figure is ten times as great as it would be if it were moved one place to the right and one-tenth as great as it would be if it were moved one place to the left, and that this is true even though the figure is moved past the decimal point.

Practice in reading and writing decimals. Decimals running to more than three or four places occur very rarely in the experiences of most persons. We may quite properly teach pupils to numerate decimals, that is, to think, "tenths, hundredths, thousandths, ten thousandths, hundred thousandths, millionths," etc., but, after all, the trained adult does not go through this formality in reading the decimals which he ordinarily encounters. On the other hand, he learns, and learns well, that one decimal place means tenths, that two decimal places mean hundredths, that three decimal places mean thousandths, etc. He may know equally well that

four decimal places mean ten thousandths or he may name over the different orders to determine what a decimal fraction of four places is called. From the very beginning of the pupil's practice in reading and writing decimal fractions, he should have his attention called to the fact that one decimal place means tenths, that two decimal places mean hundredths, and that three decimal places mean thousandths. These facts should be thoroughly memorized.

When writing decimal fractions from dictation, then, the pupil should learn to think first of the number of decimal places in the fraction stated, then of the number of digits needed to express the number which represents that fraction, and, finally, whether zeros are necessary. In writing the fraction, twenty-three thousandths, for example, he will think: "thousandths, three figures; twenty-three, two figures; need one zero"; and write promptly .023, writing the zero first rather than last. Practice will make habitual such a reaction if the practice is sufficiently abundant and is limited to decimal fractions of common occurrence. If he is required to write, *twenty-three thousand four hundred six ten-millionths,* he may have to write 23406 first and then name over the orders to determine that two zeros follow the decimal point. But we are not justified in requiring pupils to write such fractions as this from dictation. In practical experience, we sometimes read such decimals but we do not write them from dictation. In reading them we usually say, furthermore, "point, zero, zero, two, three, four, zero, six" (or, "point, O, O, two, three, four, O, six") and not, "twenty-three thousand four hundred six ten millionths" for the decimal, .0023406.

Indeed, writing decimal fractions from dictation is seldom done except in schoolrooms. Decimal fractions are read and studied and they are obtained as the result of division, etc., but they are not often written from dictation. Some practice in writing decimals from dictation is to be recommended for the aid which the experience furnishes in developing a better understanding of decimals but this practice should be limited to fractions of common occurrence and should not be continued beyond the point where it ceases to help in the development of such an understanding.

Some teachers have pupils read such a decimal as .437 as "point, four, three, seven" from the very first and when they read decimals for the pupils to write, they read them in the same manner. This plan denies the pupils part of their opportunity to learn the meaning of decimals. Eventually, decimals may be so read, as, indeed, they usually are in the laboratory, the shop, etc., but for awhile the pupils should also learn to think and to say, "four hundred thirty-seven thousandths," for .437, and "seventy-five hundredths" for .75.

In reading decimal mixed numbers, the necessity of saying "and" where the point is and nowhere else should be pointed out. Pupils who have been properly taught in reading integers have learned not to use the word "and." It is well to point out that people used to say, "four and twenty,"[1] but that today we not only

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[1] As in the nursery rime,

> Sing a song of six pence,
> A pocketful of rye,
> *Four and twenty* blackbirds
> Were baked into a pie. Etc.

turn the words around but also we drop *and*. Likewise, we say, "one hundred six," not "one hundred and six." To help pupils see the importance of using *and* in the proper place, it is well to dictate a few numbers like the following.

Six hundred twenty-four thousandths.

Six hundred *and* twenty-four thousandths.

Note that the first is written, .624, but that the second is written, 600.024. The author has heard teachers scold pupils who wrote 700.029 instead of .729 when the number dictated was, "seven hundred and twenty-nine thousandths."

The pupil's early work in reading and writing decimals should be varied and rich in its contacts with the world of practical affairs. However well the subject may be taught, we must expect the pupil's understanding of decimal fractions, like his understanding of common fractions and other topics, to grow gradually and slowly. Numerous and varied experiences with decimals will help the pupils greatly in learning what they mean. Railway time tables, sign boards, and automobile odometers have been mentioned. Baseball statistics include so called per cents which have been calculated to three decimal places. Crop reports, temperature and rainfall records, speed records made in automobile races, etc., furnish further opportunities for motivating the study of decimal fractions. The effective teacher will bring such material into the classroom where first-hand acquaintance on the part of the pupils may be assured.

Of course, money is an ever valuable means of showing the meaning of decimals. Years before they study

decimal fractions, pupils are taught to read amounts of money. In this, they have learned to say, "and," where the decimal point is. Thus, they read $24.32 as "twenty-four dollars *and* thirty-two cents."

The function of zeros in decimal fractions. Early in their experiences with decimal fractions, pupils have to learn that the effect of a zero may not be the same as it is with integers. They learn that it may be necessary to write one or more zeros after the decimal point and before the other figures of the fraction. This is contrary to the practice to which they have become accustomed in writing integers. On the other hand, they have learned that annexing a zero to an integer makes the number ten times as large as it was. Thus, 40 is ten times 4; 400 is ten times 40; etc. Now, they learn that annexing a zero to a decimal fraction does not change the value of the fraction, arithmetically. Experience with fractions representing money easily shows them that this is true. They see that four dimes represents the same amount of money as forty cents and, hence, that .4 of a dollar is equal to .40 of a dollar. They can also discover this by writing such a decimal fraction as .70 as a common fraction, $70/100$, and reducing it to tenths, getting $7/10$. As they work with decimal fractions, pupils should come to see that inserting a zero after the decimal point in a decimal fraction makes the value of the fraction one-tenth of what it was, that inserting two zeros divides the fraction by 100, etc.

Of course, pupils will learn later that annexing zeros to a decimal fraction indicates that a measurement or a quantity is accurate to a greater number of decimal

places than if the zeros are not used. Thus, if a report is made that a car averages 15.5 miles on a gallon of gasoline, it means that the average number of miles per gallon has been found to the nearest tenth. To two decimal places, the average number of miles per gallon may be anything from 15.45 to 15.54. If the report gives 15.50 as the average number of miles per gallon, however, it means that the average has been found to the nearest hundredth. To three decimal places, the average may be anything from 15.495 to 15.504. Teaching the significance of zeros in such a situation as this may well be left to the junior high school years.

Changing common fractions to decimal fractions. The operations with fractions are often much easier if decimal fractions rather than common fractions are used. Suppose, for example, that one has the mixed numbers shown to add. Since the least common denominator is 120, the example is rather difficult in common fraction form. But if each fraction is changed to a decimal, the work is decidedly easier. We are using three decimal places in order that we may have a sum correct to two places. Changing $17\frac{59}{120}$ to a mixed decimal, we have 17.492, to the nearest thousandth, which, in this case, happens to be what we obtained by using decimals. To two places, we can be assured that the sum, 17.49, is correct.

$4\frac{2}{3}$	4.667
$3\frac{1}{2}$	3.500
$1\frac{3}{4}$	1.750
$2\frac{1}{5}$	2.200
$5\frac{3}{8}$	5.375
$17\frac{59}{120}$	17.492

When they have learned decimal fractions, the pupils should be encouraged to use them. Many persons, both

pupils and adults, do not use decimals as much as they should. Computations which yield ungainly common fractions are too often allowed to stand when the approximately equivalent decimal form of expression would not only be easier to manipulate but also would be more meaningful. The author recalls a striking instance of this in monthly reports of enrollment and attendance submitted by teachers. The report blanks called for the aggregate days of attendance and the average daily attendance, the latter being obtained by dividing the former by the number of days which the school was in session in the month. Due to holidays, the number of days the school is in session is sometimes 19. Several teachers made out the report so that the item under consideration appeared as follows:

AVERAGE DAILY ATTENDANCE

Boys	Girls	Total
$21\frac{13}{19}$	$18\frac{10}{19}$	$40\frac{4}{19}$

Such fractions are almost meaningless; they are monstrous. Ordinarily, they should not be tolerated in teachers' reports or elsewhere. When converted into decimals, they are much more meaningful. They can be compared more easily with other fractions which originally had different denominators. This item would then appear as follows:

AVERAGE DAILY ATTENDANCE

Boys	Girls	Total
21.7	18.5	40.2

Of course, $40\frac{4}{19}$ is an exact statement while 40.2 is an approximation. But if a greater degree of precision is required, the computations may be carried to two or more decimal places. And, of course, if but one decimal place is used, the result should be expressed to the *nearest* tenth; if two places are used, the result should be expressed to the *nearest* hundredth; etc.

In Chapter 4, it was indicated that fractions whose denominators are 9, 11, 13, 17, etc., very rarely occur and that when they do occur they should be changed to decimals. Certainly, no well taught pupil will attempt to add 9ths, 11ths, and 13ths as common fractions (the least common denominator is 1287) but, if such fractions must be added, will convert them to decimals. The illustrations which have been given are extreme, to be sure, but the argument applies, although with less force, to many fractions of rather frequent occurrence. It is clear, then, that pupils must be skilled in changing common fractions to decimal fractions.

Teaching children to change common fractions to decimal fractions. Changing a common fraction to a decimal fraction is a very simple and straightforward procedure. It consists simply of dividing the numerator (with zeros annexed) by the denominator and placing a decimal point before the proper figure of the quotient. But an explanation of the process should make children understand it better and should reduce the probability of mistakes.

The easiest approach seems to be through the use of halves and fifths. The equivalance of $\frac{1}{2}$ and $\frac{5}{10}$, of $\frac{1}{5}$ and $\frac{2}{10}$, of $\frac{2}{5}$ and $\frac{4}{10}$, of $\frac{3}{5}$ and $\frac{6}{10}$, and of $\frac{4}{5}$ and $\frac{8}{10}$ is easily seen. The equivalance of most of

these has already been seen by changing fractions to least common denominator form and by reducing them to lowest terms. Also, pupils can be asked to think of the number of cents in $\frac{1}{2}$ of a dime, $\frac{1}{5}$ of a dime, etc. Since there are five cents in $\frac{1}{2}$ of a dime, $\frac{1}{2} = \frac{5}{10} = .5$; since there are two cents in $\frac{1}{5}$ of a dime, $\frac{1}{5} = \frac{2}{10} = .2$; etc. The same fractions can as easily be changed to hundredths. Since it is known that $\frac{1}{2}$ of a dollar is 50 cents, $\frac{1}{2} = \frac{50}{100} = .50$. Also $\frac{1}{5} = \frac{20}{100} = .20$, etc.

After this experience with halves and fifths, it is best to work for awhile on other *unit* fractions. Let us change several unit fractions having denominators less than ten to tenths. To change $\frac{1}{2}$ to tenths, we think, "In 1 there are $\frac{10}{10}$, so in $\frac{1}{2}$ there are $\frac{1}{2}$ of $\frac{10}{10}$ or $\frac{5}{10}$; then $\frac{1}{2} = .5$." To change $\frac{1}{5}$ to tenths, we think, "In 1 there are $\frac{10}{10}$, so in $\frac{1}{5}$ there are $\frac{1}{5}$ of $\frac{10}{10}$ or $\frac{2}{10}$; then $\frac{1}{5} = .2$." Now, to change $\frac{1}{3}$ to tenths, we think in the same manner, "In 1 there are $\frac{10}{10}$, so in $\frac{1}{3}$ there are $\frac{1}{3}$ of $\frac{10}{10}$ or $3\frac{1}{3}$ tenths; then $\frac{1}{3} = .3\frac{1}{3}$." A similar line of reasoning is followed for $\frac{1}{4}$ and other unit fractions.

Next, change a few unit fractions to hundredths. Thus, "In 1 there are $\frac{100}{100}$, so in $\frac{1}{4}$ there are $\frac{1}{4}$ of $\frac{100}{100}$ or $\frac{25}{100}$; then $\frac{1}{4} = .25$. Also, "In 1 there are $\frac{100}{100}$, so in $\frac{1}{6}$ there are $\frac{1}{6}$ of $\frac{100}{100}$ or $16\frac{2}{3}$ hundredths; then $\frac{1}{6} = .16\frac{2}{3}$. We have written "$3\frac{1}{3}$ tenths" and "$16\frac{2}{3}$ hundredths" instead of $\dfrac{3\frac{1}{3}}{10}$ and $\dfrac{16\frac{2}{3}}{100}$ to avoid the confusing form of complex fractions. Of course, each of these decimals which represent uneven divisions can be expressed to the nearest tenth or to the nearest hundredth and without fractional remainders.

Thus it is seen that any unit fraction can be changed to a decimal by annexing zeros to the numerator and then dividing by the denominator. But since $\frac{1}{5} = .2$, $\frac{2}{5}$ must be two times .2, or .4; since $\frac{1}{4} = .25$, $\frac{3}{4}$ must be three times .25, or .75; etc. Then, to change any common fraction to a decimal we simply annex zeros to the numerator and then divide by the denominator.

The brighter pupils probably will succeed in following the line of reasoning which has just been sketched; the average may succeed; the duller will not. Here, as elsewhere in explaining new processes in arithmetic, we should adapt the nature and extent of the explanation to the varying capacities of the pupils. As much explanation as can be appreciated and understood should be given. If we do no more than call attention to the fact that $\frac{1}{2}$ is the same as $\frac{5}{10}$, or .5, and that we can get the same result by annexing a zero to the numerator and then dividing by the denominator we have done something toward rationalizing the process and have made it more meaningful to the pupils to just that extent. However complete the explanation, the class will soon arrive at the summary rule, To change a common fraction to a decimal, divide the numerator by the denominator, annexing as many zeros as necessary. This will be followed by the discovery that the number of decimal places will be the same as the number of zeros annexed to the numerator.

If the denominator is larger than 10 and the numerator is a very small number, the example is much harder for some pupils due to the fact that numerator can not be divided by the denominator until two or more zeros have been annexed. In changing the fraction, $\frac{1}{12}$, to a

decimal, for example, pupils sometimes get .833 instead of .083. There should be considerable practice on examples of this type. To construct examples of this type, the teacher should remember that the numerator must be 1 if the denominator is in the 'teens, the numerator must be 1 or 2 if the denominator is in the twenties, it must be 1, 2, or 3 if the denominator is in the thirties, etc. The pupil should see that in converting proper fractions to decimals, something must be written in the quotient every time a zero is annexed to the numerator. If nothing else can be written in the quotient, a zero is written there.

Sometimes, the pupil can easily check his work by writing the decimal which he obtains as a common fraction, reducing to lowest terms, and comparing the result with the fraction with which he started. Thus, if he has changed $1/250$ to a decimal and has obtained .04, he will write, $.04 = 4/100 = 1/25$. Thus, his error is revealed. Fractions which yield uneven divisions can also be checked in this manner but the check is more difficult to apply. Thus, $1/12 = .08\frac{1}{3}$. Then, $.08\frac{1}{3} = \dfrac{8\frac{1}{3}}{100} =$

$$\frac{25}{3} \div 100 = \frac{\overset{1}{\cancel{25}}}{3} \times \frac{1}{\underset{4}{\cancel{100}}} = \frac{1}{12}.$$

The decimal equivalents of some common fractions should be memorized. The constant use of these means that the pupils gradually memorize them but, eventually, there should be a minimum list which every pupil will know. There is difference of opinion as to how many are worth the time and trouble required for pupils to memorize them but it seems that we may in-

clude the following in a minimum list. These occur frequently enough to justify us in having their decimal equivalents learned as a means of facilitating computations in many of the topics which are studied in arithmetic.

$\frac{1}{2} = .5$	$\frac{1}{5} = .2$	$\frac{5}{6} = .833$
$\frac{1}{4} = .25$	$\frac{2}{5} = .4$	$\frac{1}{8} = .125$
$\frac{3}{4} = .75$	$\frac{3}{5} = .6$	$\frac{3}{8} = .375$
$\frac{1}{3} = .333$	$\frac{4}{5} = .8$	$\frac{5}{8} = .625$
$\frac{2}{3} = .667$	$\frac{1}{6} = .167$	$\frac{7}{8} = .875$

Pupils' errors in decimals. Brueckner gave tests in decimals to more than 300 pupils in Grades VI, VII, and VIII in four schools in Minneapolis. His tests were in two parts. Part 1 represented an attempt to measure the ability of pupils to read, write, and convert decimals, and their understanding of the value of decimals. Part 2 consisted of diagnostic tests in each of the four fundamental operations.[2]

It was discovered that these pupils lacked much of having an adequate understanding of the numerical values of decimals. This was shown by their inability to arrange a series of decimal mixed numbers in order of their size. Their next most serious weakness, as shown by this part of the test, was in their ability to spell the words required in writing decimals in words.

In all four of the fundamental operations, the outstanding errors were errors in computation and errors in placing the decimal point. The decimal point often was misplaced but in many cases it was omitted. There

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[2] Brueckner, Leo J. *Diagnostic and Remedial Teaching in Arithmetic.* Philadelphia: The John C. Winston Company, 1930, pp. 219-238.

were other errors which had to do with the use of zeros, changing common fractions to decimals, etc.

The total number of errors was 580 in addition, 465 in subtraction, 1,814 in multiplication, and 3,751 in division. It is apparent from this study that these pupils need a better basic understanding of the meaning of decimals, that they need greater degrees of skill in the fundamental operations with integers, and that they need better teaching of the operations with decimals, especially in multiplication and division.

Addition and subtraction of decimal fractions. Pupils who are competent in the addition and subtraction of integers should not find it difficult to add and subtract decimals. If they keep the decimal points in a straight column and are careful to write tenths under tenths, hundredths under hundredths, etc., as well as to write units under units, tens under tens, etc., and if they can add and subtract, they should soon be able to add and subtract decimal fractions and decimal mixed numbers as well as integers.

However, Brueckner found that pupils had difficulty in placing the decimal point in the sum, especially in examples such as those shown herewith.[8] These are examples 2, 4, and 6 in his test in the addition of decimals. The specific nature of the difficulty here is not stated but the author has observed that some pupils will write .16, instead of 1.6, as the sum in (a). Since the addends are all fractions, they tend to make the sum a fraction even when it is a mixed number. In (b) a common an-

(a)	(b)	(c)
.3	.28	.05
.5	.43	.09
.8	.95	.08

[8] Brueckner, Leo J., *op. cit.*, pp. 221 and 231.

swer for such pupils is .166, instead of 1.66, and in (c), .022, instead of .22.

In subtraction, the pupils are sometimes disturbed if the number of decimal places in the subtrahend is greater than the number of decimal places in the minuend. In such an example as that shown, for instance, the pupil must begin by subtracting from an imagined zero. Since the zeros are not there and the pupils are confused, teachers often have them fill in the blank spaces with zeros as shown. This visual aid seems quite helpful in early lessons but it should not be necessary for pupils to use it always. As pupils mature and gain in proficiency in arithmetic they should, to a greater and greater extent, become independent of such visual aids as this.

$$\begin{array}{r} 62.4 \\ \underline{13.291} \end{array}$$

$$\begin{array}{r} 62.400 \\ \underline{13.291} \end{array}$$

In early lessons in adding and subtracting decimals, it is well to show again the analogy to common fractions. Thus, $\frac{1}{10} + \frac{3}{10} = \frac{4}{10}$, or $.1 + .3 = .4$; and $\frac{8}{10} - \frac{5}{10} = \frac{3}{10}$, or $.8 - .5 = .3$. Likewise, $\frac{23}{100} + \frac{41}{100} = \frac{64}{100}$, or $.23 + .41 = .64$; and $\frac{95}{100} - \frac{38}{100} = \frac{57}{100}$, or $.95 - .38 = .57$. As a special case of this last form, we may note that $\frac{16}{100} - \frac{12}{100} = \frac{4}{100}$, or $.16 - .12 = .04$, and show that it is necessary to write the zero before the 4, a practice which is not followed in subtraction with integers.

In connection with such work with common fractions, we may anticipate and thereby avoid some of the difficulties which Brueckner discovered. For example, $\frac{3}{10} + \frac{7}{10} + \frac{7}{10} = \frac{17}{10} = 1\frac{7}{10}$, or $.3 + .7 + .7 = 1.7$ (rather than .17). Again, $\frac{53}{100} + \frac{21}{100} + \frac{37}{100} =$

$^{111}/_{100} = 1^{11}/_{100}$, or $.53 + .21 + .37 = 1.11$ (rather than .111).

In their early work with decimal fractions, pupils often find it difficult to tell which of two fractions is the larger. Thus, they may conclude that .3 is smaller than .16. Of course, this indicates a lack of understanding of place value in decimal fractions and the need for better and more thorough instruction on the fundamental meaning of decimals. But even with good instruction there should be opportunities for practice in comparing two or more decimal fractions as to size. Exercises such as the following may be used.

Arrange these sets of decimal fractions in the order of their size, putting the largest of each set first, the next largest second, and so on to the smallest which should be placed last.

.32	.125	.063	.5	.009
.8	.047	.71	.05	.214
.116	.214	.002	.43	.5
.1	.001	.01	1.0	1.01

Comparisons may also be made in subtraction with decimals by stating the examples in the following form.

Find the difference between 4.2 and 2.9; between .7 and .24; between .044 and .22; between 1.31 and 1.8; etc. The pupil must decide which of the two fractions of a pair to use as minuend.

MULTIPLICATION WITH DECIMALS

Multiplication of a decimal by an integer. I will be noted that we are grouping examples in multiplication with decimals into three main classes and that these

classes correspond closely to the three major classes recognized in multiplication with common fractions. It will be seen that the same thing is true for division with decimals.

Problems which require that a decimal be multiplied by an integer are easily found. The teacher may use a problem and develop the subject as follows:

TEACHER: "One day, while riding in the front seat with his father, Jack noticed that they drove .7 of a mile in a minute. At this rate, how far would they go in 5 minutes? How shall we find out?"

PUPIL: "Multiply .7 by 5."

TEACHER: "We can write the example as we usually write multiplication examples. If we multiply 7 by 5, we get 35. But the 7 means tenths; so we get 35 what?"

$$\begin{array}{r} .7 \\ \underline{5} \\ 3.5 \end{array}$$

PUPIL: "Thirty-five tenths."

TEACHER: "But 35 tenths is the same as 3.5. Let us solve the example in another way. We can write .7 as a common fraction. Then, $5 \times \frac{7}{10} = \frac{35}{10} = 3\frac{5}{10} = 3.5$."

As other examples of this kind are solved, the pupils see that if tenths are multiplied, the product is tenths but that if there are more than 10 tenths in the product, the product is a decimal mixed number. The examples will become gradually more elaborate. The multiplicands will be two- and three-place decimals. They will also be decimal mixed numbers. The multiplier may be a number of one or more digits. After a time pupils discover that the number of decimal places in a product is always the same as the number of decimal places in the multiplicand when the multiplier is an integer.

Practice Examples in Decimal Fractions, Set 1, illus-

trate the various types of examples which may be used for practice after each of these types has been presented. It will be seen that these examples include a rather wide variety of types. No type occurs twice in the set. Each of the 45 primary multiplication combinations occurs one or more times in the set. None occurs more than three times.

We have a special case of multiplication of a decimal fraction by an integer when we multiply by an integral power of 10—10, 100, 1000, etc. Two or three illustrative examples will soon lead the pupils to see that when we multiply by 10, we simply move the decimal point one place to the right; when we multiply by 100, we move the point two places to the right; etc. In general, when we multiply by a power of 10, we move the decimal point as many places to the right as there are zeros in the multiplier. The teacher should provide sufficient practice on examples of this type to enable the pupils to multiply decimals by powers of 10 easily and quickly.

Practice Examples in Decimal Fractions. Set 1

Multiplication of a Fraction by an Integer

.24 3	1.6 4	.5 9	14.6 7	15.28 5	243.6 9
647.28 8	6.247 2	35.768 3	6.45 6	981.753 9	
.746 5	.26 24	73.21 30	4.972 47	.148 80	
628.9 19	5.01 70	15.4 31	87.603 69	7.8 37	

Multiplication of an integer by a decimal. In their early work in multiplication with integers, pupils are taught to use as multiplier the number which it is the more convenient to use as multiplier, regardless of any distinctions between abstract and concrete numbers (Chapter 3). In other words, they have learned that the multiplier and the multiplicand can be interchanged without affecting the value of the product. Indeed, they have been taught to interchange them as one of the checks employed in multiplication, if the two numbers do not differ greatly in the number of digits they contain.

Multiplication of an integer by a decimal, then, should be treated along with multiplication of a decimal by an integer as simply another phase of the same general type of work. It is well to let the pupils understand at once that there are no new difficulties involved. The number of decimal places in the product will be the same as the number in the multiplier just as the number in the product has been the same as the number in the multiplicand. If necessary, the procedure can be rationalized by reverting to common fractions as in the case of multiplication of a decimal by an integer. Thus $.4 \times 7 = \frac{4}{10} \times 7 = \frac{28}{10} = 2\frac{8}{10} = 2.8$. Examples for practice similar to those in Set 1 are easily prepared. The only difference will be that the decimal expression will occur in the multiplier instead of in the multiplicand.

In multiplying an integer by a decimal, it is well to review a principle which was emphasized in Chapter 6 in the discussion of multiplication with common frac-

tions, *viz.*, that multiplying a number by a fraction is equivalent to finding a part of that number. Thus,

$$.2 \times 12 = .2 \text{ of } 12 = 2.4$$
$$.7 \times 8 = .7 \text{ of } 8 = 5.6$$
$$.4 \text{ of } 20 = .4 \times 20 = 8.0$$
$$.15 \text{ of } 25 = .15 \times 25 = 3.75, \text{ etc.}$$

Multiplication of a decimal by a decimal. So far, the pupil has learned to solve examples in which either the multiplicand or the multiplier contained decimal fractions and he has learned that the number of decimal places in the product is the same as the number in the multiplicand or the number in the multiplier. But he has not learned how to solve examples in which both the multiplicand and the multiplier contain decimal fractions.

The best approach to this third group of examples in multiplication with decimals seems to be through the medium of common fractions. The illustrative examples which follow can be used in developing the subject before a class. In each case, a decimal example is stated and then solved as an example in common fractions. Finally, the decimal solution is performed.

(*a*) $\begin{array}{r} .3 \\ \underline{.7} \end{array} = \frac{7}{10} \times \frac{3}{10} = \frac{21}{100} = .21 \qquad \begin{array}{r} .3 \\ \underline{.7} \\ .21 \end{array}$

(*b*) $\begin{array}{r} .3 \\ \underline{.3} \end{array} = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100} = .09 \qquad \begin{array}{r} .3 \\ \underline{.3} \\ .09 \end{array}$

(*c*) $\begin{array}{r} .23 \\ \underline{.7} \end{array} = \frac{7}{10} \times \frac{23}{100} = \frac{161}{1000} = .161 \qquad \begin{array}{r} .23 \\ \underline{.7} \\ .161 \end{array}$

(*d*) $\begin{array}{r} .19 \\ \underline{.3} \end{array} = \frac{3}{10} \times \frac{19}{100} = \frac{57}{1000} = .057 \qquad \begin{array}{r} .19 \\ \underline{.3} \\ .057 \end{array}$

(*e*) $\begin{array}{r} .03 \\ \underline{.3} \end{array} = \frac{3}{10} \times \frac{3}{100} = \frac{9}{1000} = .009 \qquad \begin{array}{r} .03 \\ \underline{.3} \\ .009 \end{array}$

(*f*) $\begin{array}{r} .37 \\ \underline{.67} \end{array} = \frac{67}{100} \times \frac{37}{100} = \frac{2479}{10000} = .2479 \qquad \begin{array}{r} .37 \\ \underline{.67} \\ 259 \\ \underline{222} \\ .2479 \end{array}$

(*g*) $\begin{array}{r} .37 \\ \underline{.23} \end{array} = \frac{23}{100} \times \frac{37}{100} = \frac{851}{10000} = .0851 \qquad \begin{array}{r} .37 \\ \underline{.23} \\ 111 \\ \underline{74} \\ .0851 \end{array}$

(*h*) $\begin{array}{r} .13 \\ \underline{.07} \end{array} = \frac{7}{100} \times \frac{13}{100} = \frac{91}{10000} = .0091 \qquad \begin{array}{r} .13 \\ \underline{.07} \\ .0091 \end{array}$

(*i*) $\begin{array}{r} .03 \\ \underline{.03} \end{array} = \frac{3}{100} \times \frac{3}{100} = \frac{9}{10000} = .0009 \qquad \begin{array}{r} .03 \\ \underline{.03} \\ .0009 \end{array}$

(*j*) $\begin{array}{r} 4.1 \\ \underline{.9} \end{array} = \frac{9}{10} \times \frac{41}{10} = \frac{369}{100} = \frac{369}{100} = 3.69 \qquad \begin{array}{r} 4.1 \\ \underline{.9} \\ 3.69 \end{array}$

(*k*) $\begin{array}{r} 4.3 \\ \underline{2.3} \end{array} = \frac{23}{10} \times \frac{43}{10} = \frac{989}{100} = \frac{989}{100} = 9.89 \qquad \begin{array}{r} 4.3 \\ \underline{2.3} \\ 129 \\ \underline{86} \\ 9.89 \end{array}$

It will be seen that in both (*a*) and (*b*) tenths are multiplied by tenths but that in (*b*) it is necessary to

insert a zero in the product. In (*c*), (*d*), and (*e*) hundredths are multiplied by tenths. In (*c*) there are no zeros in the product, in (*d*) there is one zero in the product, and in (*e*) there are two zeros in the product. Examples (*f*), (*g*), (*h*), and (*i*) are alike in that in each hundredths are multiplied by hundredths. In (*f*) there are no zeros in the product but all possible numbers of zeros are found in the products of (*g*), (*h*), and (*i*). In (*j*) a mixed number is multiplied by a fraction and in (*k*) a mixed number is multiplied by a mixed number. In each of these illustrative examples, the numbers have been chosen so that reducing to lowest terms (or canceling) will not be possible. Of course, it is necessary that the denominator of the product shall always be a power of 10.

Many more variations can be introduced in examples of this kind. We have not included thousandths in either the multiplicands or the multipliers. We have given but two examples involving decimal mixed numbers and in neither of these is there more than one decimal place in the multiplicand or in the multiplier. However, enough illustrative examples have been given to show the pupil that *when a decimal is multiplied by a decimal the number of decimal places in the product is equal to the number in the multiplicand plus the number in the multiplier.*

Space will not be taken to present sets of practice examples in the multiplication of a decimal by a decimal. In preparing such material the teacher should include the eleven types of examples which have been illustrated and should provide also for the multiplication of thousandths by tenths, thousandths by hundredths,

thousandths by thousandths, etc. Each of the first nine types should appear in mixed numbers as well as fractions alone.

DIVISION WITH DECIMALS

In Chapter 6, it was observed that division is the hardest of the four fundamental operations with common fractions. Likewise, it is the hardest of the operations with decimal fractions. Of course, the difficulty lies in locating the decimal point in the quotient. If this matter can be explained so that the pupils understand it and if a sufficient amount of the proper kind of practice is provided, pupils should be able to divide with decimal fractions as well as with integers.

It is essential that the work be graded so that, so far as possible, but one difficulty is encountered at a time. A proper sequence of examples will do much toward making the subject easier for pupils to understand and learn. We shall use the same three major types of examples that have been used in multiplication and division with common fractions and in multiplication with decimals.

Division of a decimal by an integer. The process of division of a decimal by an integer may be rationalized by a method very similar to that used in rationalizing the process of multiplication of a decimal by an integer. If the pupil is to divide .8 by 4, for example, he should see that this is like dividing 8 by 4 except that the 8 represents tenths instead of ones or units. Hence, dividing .8 by 4 is like dividing 8 apples by 4. The quotient, 2, means 2 tenths just as the quotient is 2 apples if 8 apples are divided by 4. Hence, $.8 \div 4 = .2$.

Common fractions may be used also to rationalize this process and to enable the pupil to decide where the decimal point should be placed in the quotient. Of course, we should begin with examples which do not have remainders. The more important steps are summarized in the following illustrations.

(*a*) $3\overline{).9} = \frac{9}{10} \div 3 = \frac{\overset{3}{\cancel{9}}}{10} \times \frac{1}{\underset{1}{\cancel{3}}} = \frac{3}{10} = .3 \qquad \begin{array}{r} .3 \\ 3\overline{).9} \end{array}$

(*b*) $7\overline{).91} = \frac{91}{100} \div 7 = \frac{\overset{13}{\cancel{91}}}{100} \times \frac{1}{\underset{1}{\cancel{7}}} = \frac{13}{100} = .13 \qquad \begin{array}{r} .13 \\ 7\overline{).91} \end{array}$

(*c*) $7\overline{).49} = \frac{49}{100} \div 7 = \frac{\overset{7}{\cancel{49}}}{100} \times \frac{1}{\underset{1}{\cancel{7}}} = \frac{7}{100} = .07 \qquad \begin{array}{r} .07 \\ 7\overline{).49} \end{array}$

(*d*) $7\overline{).861} = \frac{861}{1000} \div 7 = \frac{\overset{123}{\cancel{861}}}{1000} \times \frac{1}{\underset{1}{\cancel{7}}} = \frac{123}{1000} = .123 \qquad \begin{array}{r} .123 \\ 7\overline{).861} \end{array}$

(*e*) $3\overline{).117} = \frac{117}{1000} \div 3 = \frac{\overset{39}{\cancel{117}}}{1000} \times \frac{1}{\underset{1}{\cancel{3}}} = \frac{39}{1000} = .039 \qquad \begin{array}{r} .039 \\ 3\overline{).117} \end{array}$

(*f*) $3\overline{).021} = \frac{21}{1000} \div 3 = \frac{\overset{7}{\cancel{21}}}{1000} \times \frac{1}{\underset{1}{\cancel{3}}} = \frac{7}{1000} = .007 \qquad \begin{array}{r} .007 \\ 3\overline{).021} \end{array}$

(*g*) $9\overline{)6.3} = \frac{63}{10} \div 9 = \frac{\overset{7}{\cancel{63}}}{10} \times \frac{1}{\underset{1}{\cancel{9}}} = \frac{7}{10} = .7 \qquad \begin{array}{r} .7 \\ 9\overline{)6.3} \end{array}$

(*h*) $9\overline{)1.89} = \frac{189}{100} \div 9 = \frac{\overset{21}{\cancel{189}}}{100} \times \frac{1}{\underset{1}{\cancel{9}}} = \frac{21}{100} = .21 \qquad \begin{array}{r} .21 \\ 9\overline{)1.89} \end{array}$

Through examples such as these the pupils are led to discover that *when the divisor is a whole number there are always just as many decimal places in the quotient as in the dividend* and that it is sometimes neces-

sary to write zeros just after the decimal point in the quotient.

In Chapter 3, it was suggested that in division with integers the pupil be taught to place each quotient figure directly above the last figure of the partial dividend being used. This is done in earlier years to help the pupil keep track of the dividend figures which have been used and to help him remember that sometimes he must place a zero in the quotient. Another reason for this orderly and systematic method of writing quotient figures will now be apparent. *In dividing a decimal fraction or a decimal mixed number by an integer the point is placed in the quotient directly above the point in the dividend.* Emphasis upon this matter will help the pupil to remember that he must place zeros in the quotient when they are needed there.

In providing for further practice in division of decimals by integers, dividends which yield remainders should be chosen. Some of these will come out even, if the division is carried far enough, but others belong to the circulating type. It is well, in dealing with such examples to instruct pupils to carry the quotient to one, two, or three decimal places and to express the quotient to the *nearest* tenth, or hundredth, or thousandth.

Pupils with more than ordinary native ability and liking for mathematics may enjoy working with circulating decimals as a recreation. A few children will be delighted with the cycles of digits which unroll before their eyes. The sevenths offer an interesting illustration. There are six digits in the cycle. Each of the sevenths which is a proper fraction is carried to 12 decimal places below, or through two cycles. Note that for each of

these fractions the sequence of digits is the same but that the sequence begins with a different digit in each case.

All pupils should see that if we change $1/3$ to a decimal we get a never-ending series of 3's; that $2/3$ yields a perpetual series of 6's; that $1/6$ gives us 1 as the first quotient figure and then an infinite number of 6's; that $5/6$ yields an 8 and limitless 3's; etc. Changing ninths and elevenths to decimals will be an interesting diversion for pupils who enjoy this sort of thing.

$$1/7 = .1\,4\,2\,8\,5\,7\,1\,4\,2\,8\,5\,7$$
$$2/7 = .2\,8\,5\,7\,1\,4\,2\,8\,5\,7\,1\,4$$
$$3/7 = .4\,2\,8\,5\,7\,1\,4\,2\,8\,5\,7\,1$$
$$4/7 = .5\,7\,1\,4\,2\,8\,5\,7\,1\,4\,2\,8$$
$$5/7 = .7\,1\,4\,2\,8\,5\,7\,1\,4\,2\,8\,5$$
$$6/7 = .8\,5\,7\,1\,4\,2\,8\,5\,7\,1\,4\,2$$

Examples for practice in the division of a decimal by an integer are given in Set 2. This set provides practice on a wide variety of examples. Divisors of one, two, and three digits are included. Dividends contain from one to five decimal places. Various forms of zero difficulties are included.

Practice Examples in Decimal Fractions. Set 2

Division of a Decimal by an Integer

$3\overline{).6}$	$2\overline{).9}$	$7\overline{).21}$	$9\overline{).65}$	$3\overline{).5}$
$4\overline{).88}$	$8\overline{).162}$	$6\overline{).0614}$	$4\overline{)24.6}$	$7\overline{)2.35}$
$9\overline{)6.36}$	$6\overline{)12.13}$	$2\overline{)271.1}$	$8\overline{)54.28}$	$5\overline{)17.29}$
$48\overline{)138.04}$	$97\overline{)5716.2}$	$12\overline{)5.518}$	$81\overline{)2.5768}$	

$59\overline{)\,.28924}$	$73\overline{)\,41.427}$	$27\overline{)\,1.2922}$	$65\overline{)\,192.83}$
$32\overline{)\,3105.6}$	$46\overline{)\,214.22}$	$53\overline{)\,47.837}$	$77\overline{)\,5.3961}$
$85\overline{)\,.47386}$	$67\overline{)\,.40349}$	$28\overline{)\,25.374}$	$95\overline{)\,.63037}$
$185\overline{)\,1103.938}$	$462\overline{)\,31.9542}$	$554\overline{)\,4038.26}$	$738\overline{)\,73261.6}$

Division of an integer by a decimal. The teacher may begin with a very simple example, such as $.3\overline{)\,9}$, and have this example solved by the method of common fractions to determine what the quotient is. Thus,

$$.3\overline{)\,9} = 9 \div \tfrac{3}{10} = \overset{3}{\cancel{9}} \times \frac{10}{\underset{1}{\cancel{3}}} = 30. \text{ So, } \overset{30}{.3\overline{)\,9}}.$$

A few more examples of the same kind—examples with one-digit divisors and no remainders—may be solved on the blackboard. These may be followed by examples having two-digit divisors and no remainders. The following examples are illustrative.

(*a*) $.3\overline{)\,9} = 9 \div \tfrac{3}{10} = \overset{3}{\cancel{9}} \times \frac{10}{\underset{1}{\cancel{3}}} = 30 \qquad \overset{30}{.3\overline{)\,9}}$

(*b*) $.7\overline{)\,28} = 28 \div \tfrac{7}{10} = \overset{4}{\cancel{28}} \times \frac{10}{\underset{1}{\cancel{7}}} = 40 \qquad \overset{40}{.7\overline{)\,28}}$

(*c*) $.3\overline{)\,87} = 87 \div \tfrac{3}{10} = \overset{29}{\cancel{87}} \times \frac{10}{\underset{1}{\cancel{3}}} = 290 \qquad \overset{290}{.3\overline{)\,87}}$

(*d*) $.9\overline{)\,495} = 495 \div \tfrac{9}{10} = \overset{55}{\cancel{495}} \times \frac{10}{\underset{1}{\cancel{9}}} = 550 \qquad \overset{550}{.9\overline{)\,495}}$

(*e*) $.13\overline{)\,26} = 26 \div \tfrac{13}{100} = \overset{2}{\cancel{26}} \times \frac{100}{\underset{1}{\cancel{13}}} = 200 \qquad \overset{200}{.13\overline{)\,26}}$

(*f*) $.21\overline{)63} = 63 \div {}^{21}\!/_{100} = \overset{3}{\cancel{63}} \times {}^{100}\!/_{\underset{1}{\cancel{21}}} = 300$ $\qquad\begin{array}{r}300\\ .21\overline{)63}\end{array}$

(*g*) $.11\overline{)121} = 121 \div {}^{11}\!/_{100} = \overset{11}{\cancel{121}} \times {}^{100}\!/_{\underset{1}{\cancel{11}}} = 1100$ $\qquad\begin{array}{r}1100\\ .11\overline{)121}\end{array}$

The solution of such examples should lead the pupils to see that when a whole number is divided by tenths, one zero must be annexed to the quotient; that when a whole number is divided by hundredths, two zeros must be annexed to the quotient; etc. In other words, when a whole number is divided by a decimal, as many zeros must be annexed to the quotient as there are decimal places in the divisor.

In Chapter 6, it was emphasized that division by a fraction is best rationalized by stressing the measurement idea of division. Thus, $4 \div {}^{1}\!/_{2} =$ has more meaning when it is read, "How many halves in 4?" This same idea helps in making meaningful the division of an integer by a decimal fraction. Thus, $.1\overline{)6}$ suggests the question, "How many tenths in 6?" Obviously, the answer to this question can not be 6. Since there are 10 tenths in 1, there must be 60 tenths in 6. Hence, $\begin{array}{r}60\\ .1\overline{)6}\end{array}$. Likewise, $.2\overline{)8}$ suggests the question, "How many .2's in 8?" Of course, there are four 2's in 8. But there are five .2's in 1; therefore, there are forty .2's in 8. Hence, $\begin{array}{r}40\\ .2\overline{)8}\end{array}$.

Another principle which was learned in dividing by a common fraction should be reviewed here also, *viz.*, *when the divisor is less than one the quotient is larger*

than the dividend. The pupil should examine the illustrative examples which have been given with this in mind. He will observe that in each case the divisor is less than one and that the quotient is larger than the dividend.

Additional illustrative examples in which there may be zeros immediately following the decimal point in the divisor should be given. The following is typical.

$$.03\overline{)15} = 15 \div \tfrac{3}{100} = \overset{5}{\cancel{15}} \times \frac{100}{\underset{1}{\cancel{3}}} = 500 \qquad \begin{array}{r} 500 \\ .03\overline{)15} \end{array}$$

Then, decimal mixed numbers should be used as divisors.

$$1.3\overline{)39} = 39 \div 1\tfrac{3}{10} = 39 \div \tfrac{13}{10} = \overset{3}{\cancel{39}} \times \frac{10}{\underset{1}{\cancel{13}}} = 30 \qquad \begin{array}{r} 30 \\ 1.3\overline{)39} \end{array}$$

The pupil sees that what he has learned holds true, that when a whole number is divided by a decimal as many zeros must be annexed to the quotient as there are decimal places in the divisor.

Further confidence in the accuracy of the newly learned procedure may be induced by employing multiplication as a check. The pupil knows that he should obtain the dividend by multiplying the divisor by the quotient. He also knows how to point off the product when a decimal has been multiplied by an integer. Thus, if 375 has been divided by 12.5 and the quotient, 30, has been obtained, it is readily seen that $30 \times 12.5 = 375$. But the product of 3 and 12.5 is only 37.5.

But after all, the best check is the check of common sense. In the same example, for instance, the pupil

should be taught to think after dividing, "How many 12's in 375? There are more than 3, for three 12's are 36. There couldn't be 300, for $300 \times 12 = 3600$. The answer must be 30." Then it is observed that $30 \times 12 = 360$, which is almost 375. Likewise, in the accompanying example the pupil should think, "37 can't be right because $4 \times 37 = 148$. The answer must be 370." Pupils do not use this check, the check of common sense, as much as they should. They may forget rules for locating the decimal point, although if the matter is explained so that they can understand it there seems to be less chance of their forgetting, but if they are taught to inspect their work and to ask themselves, "Is the answer a sensible answer?" many of the ludicrous blunders which they make in division with decimals should be avoided.

$$\begin{array}{r} 370 \\ 4.2\overline{)1554} \\ \underline{126} \\ 294 \\ \underline{294} \end{array}$$

The pupils should proceed next to examples yielding remainders. At first, these remainders may be those which cause the examples to come out even if one or more zeros are annexed to the dividend. Then those producing non-terminating decimals may be included. Ordinarily, pupils should be instructed to express the quotients to the nearest hundredth or the nearest thousandth. The following examples are typical.

(*a*) $.2\overline{)7} = 7 \div \frac{2}{10} = 7 \times \frac{10}{2} = \frac{70}{2} = 35$ $\qquad \begin{array}{r} 35 \\ .2\overline{)7.0} \end{array}$

(*b*) $.2\overline{)9} = 9 \div \frac{2}{10} = 9 \times \frac{10}{2} = \frac{90}{2} = 45$ $\qquad \begin{array}{r} 45 \\ .2\overline{)9.0} \end{array}$

(*c*) $.4\overline{)5} = 5 \div \frac{4}{10} = 5 \times \frac{10}{4} = \frac{50}{4} = 12.5$ $\qquad \begin{array}{r} 12.5 \\ .4\overline{)5.00} \end{array}$

$$(d)\quad .8\overline{)3} = 3 \div \tfrac{8}{10} = 3 \times \tfrac{10}{8} = \tfrac{30}{8} = 3.75 \qquad \begin{array}{r} 3.75 \\ .8\overline{)3.000} \end{array}$$

$$(e)\quad .3\overline{)7} = 7 \div \tfrac{3}{10} = 7 \times \tfrac{10}{3} = \tfrac{70}{3} = 23.33 \qquad \begin{array}{r} 23.33 \\ .3\overline{)7.000} \end{array}$$

$$(f)\quad .7\overline{)6} = 6 \div \tfrac{7}{10} = 6 \times \tfrac{10}{7} = \tfrac{60}{7} = 8.57 \qquad \begin{array}{r} 8.57 \\ .7\overline{)6.000} \end{array}$$

From his experience with such examples the pupil discovers that he can place a decimal point after the last digit of the dividend and annex as many zeros as there are decimal places in the divisor without having decimal places in the quotient, but that for every additional zero that he annexes to the dividend he will have a decimal place in the quotient. Thus, in examples (*a*) and (*b*) we annex one zero to the dividend and have no decimal places in the quotient. But in example (*c*) we annex two zeros to the dividend and, therefore, have one decimal place in the quotient. Likewise, in example (*d*) we annex three zeros to the dividend and have two decimal places in the quotient. In examples (*e*) and (*f*) we can annex as many zeros to the dividend as we please but the quotient will never come out even. We have annexed three zeros to the dividend in each of these examples and have two decimal places in the quotient. The pupil should further verify the correctness of these solutions by multiplying the quotient by the divisor and adding in the remainder, if any, to secure the dividend.

Examples in which there are two or three digits in either divisor or dividend or both will be used for further practice. In some cases the divisor should be a decimal mixed number.

There should be much practice in dividing integers by decimals, particularly with dividends which yield remainders, for this latter type paves the way to the division of a decimal by a decimal. It is not necessary to use space for sets of practice examples for the teacher can easily prepare them. Each of the types which has been illustrated or described in this section should occur several times.

Division of a decimal by a decimal. The most difficult examples in division with decimals occur when there are decimal fractions in both the divisor and the dividend. Of course, the difficulty lies in deciding where to place the decimal point in the quotient, since pupils already know how to divide and they have had experience in annexing zeros to the dividend. It is easy to hand the pupils ready-made rules, such as, "The number of decimal places in the dividend minus the number in the divisor equals the number in the quotient" and "If there is a greater number of decimal places in the divisor than in the dividend, annex zeros until the number of places in the dividend equals the number in the divisor." Pupils learn such rules readily enough and the immediate results are likely to be gratifying. But the results of such instruction are not likely to be permanent. Pupils have an annoying way of forgetting rules, especially rules which have been learned by rote. And if they forget a rule whose origin is obscure to them, they are helpless. They will probably guess and they may guess wrong.

If instruction in decimals is to be based upon the meaning theory, rules for locating decimal points will be discovered by the pupils through experiences which

they understand. Again, it should be emphasized that although skill in arithmetic is worthwhile it is less to be desired than skill with understanding for the latter has the advantage of being far more educative.

Again, the best approach seems to be through the medium of common fractions. A few illustrative examples will be helpful.

(a) $.2\overline{).4} = \frac{4}{10} \div \frac{2}{10} = \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{10}}} \times \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{2}}} = 2$ $\qquad \begin{array}{r} 2 \\ .2\overline{).4} \end{array}$

(b) $.2\overline{).04} = \frac{4}{100} \div \frac{2}{10} = \frac{\overset{2}{\cancel{4}}}{\underset{10}{\cancel{100}}} \times \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{2}}} = \frac{2}{10} = .2$ $\qquad \begin{array}{r} .2 \\ .2\overline{).04} \end{array}$

(c) $.2\overline{).004} = \frac{4}{1000} \div \frac{2}{10} = \frac{\overset{2}{\cancel{4}}}{\underset{100}{\cancel{1000}}} \times \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{2}}} = \frac{2}{100} = .02$ $\qquad \begin{array}{r} .02 \\ .2\overline{).004} \end{array}$

(d) $.02\overline{).004} = \frac{4}{1000} \div \frac{2}{100} = \frac{\overset{2}{\cancel{4}}}{\underset{10}{\cancel{1000}}} \times \frac{\overset{1}{\cancel{100}}}{\underset{1}{\cancel{2}}} = \frac{2}{10} = .2$ $\qquad \begin{array}{r} .2 \\ .02\overline{).004} \end{array}$

(e) $.02\overline{).04} = \frac{4}{100} \div \frac{2}{100} = \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{100}}} \times \frac{\overset{1}{\cancel{100}}}{\underset{1}{\cancel{2}}} = 2$ $\qquad \begin{array}{r} 2 \\ .02\overline{).04} \end{array}$

The teacher will review the fact that in multiplication with decimals the number of decimal places in the product is equal to the number in the multiplicand plus the number in the multiplier. Then, since division is checked by multiplication and the dividend is equal to the divisor times the quotient, the number of decimal places in the dividend must be equal to the number in the divisor plus the number in the quotient. So, the number of decimal places in the quotient must be the number in the dividend less the number in the divisor. The teacher will develop these points while making fre-

quent reference to the illustrative examples which have been worked out on the blackboard.

The first examples which are given to pupils for practice should have no remainders, no serious difficulties in division as such, and they should not require the addition of zeros to the dividends. After pupils have become proficient in solving such relatively simple and easy examples, the difficulty of the practice material should be gradually increased. Special attention should be given to those examples in which the number of decimal places in the divisor is greater than the number in the dividend. Pupils have had an introduction to examples of this type when they learned to divide integers by decimal fractions. When examples yielding remainders have been introduced, pupils should be instructed as to how far the division is to be carried. Thereafter, most of the examples should yield remainders.

Practice Examples in Decimals, Set 3, illustrate the varied form which such examples may take. The divisors and dividends have one, two, and three decimal places and occur both as pure decimals and as decimal mixed numbers. A few cases of division of a decimal by an integer and of division of an integer by a decimal are included in this set. The examples of this set are illustrative only. Additional examples of each type should be prepared and used.

Each of the types of examples which have been described and illustrated should be represented in miscellaneous sets of examples. Miscellaneous examples constitute the real test of a pupil's ability to solve examples in division with decimal fractions. Here he must

be able to recognize the various types as he meets them and to decide in each case where the decimal point should be placed in the quotient.

When pupils hesitate or make errors in placing the decimal point the fundamental principle, *Divisor* × *Quotient* = *Dividend,* should be recalled and from this the rule for locating the decimal point should be derived. This rule may take either of two forms—*The number of decimal places in the divisor plus the number in the quotient equals the number in the dividend,* or *The number of decimal places in the dividend minus the number in the divisor equals the number in the quotient.* In either case, they must see that if the number of decimal places in the divisor exceeds the number in the dividend, zeros must be annexed to the dividend to

Practice Examples in Decimals. Set 3

Division of a Decimal by a Decimal

$.3\overline{)\,.6}$	$.4\overline{)\,.12}$	$.8\overline{)\,.344}$	$.07\overline{)\,.637}$
$.05\overline{)\,.025}$	$.06\overline{)\,.6}$	$.009\overline{)\,8.1}$	$.2\overline{)\,2.2}$
$.4\overline{)\,.13}$	$.05\overline{)\,2.6}$	$.8\overline{)\,65.3}$	$.03\overline{)\,.05}$
$.12\overline{)\,1.44}$	$.15\overline{)\,.225}$	$.25\overline{)\,62.5}$	$.44\overline{)\,2.948}$
$.042\overline{)\,.3187}$	$.027\overline{)\,132.4}$	$.143\overline{)\,.6279}$	$.568\overline{)\,20.437}$
$2.3\overline{)\,92}$	$7.4\overline{)\,345.6}$	$16.5\overline{)\,32.74}$	$2.14\overline{)\,38.91}$
$9.04\overline{)\,48.56}$	$21.8\overline{)\,6.753}$	$2.01\overline{)\,14}$	$3.19\overline{)\,264.6}$
$.75\overline{)\,.4375}$	$.68\overline{)\,68}$	$.246\overline{)\,26.34}$	$.625\overline{)\,50}$
$68\overline{)\,39.24}$	$.045\overline{)\,225}$	$96\overline{)\,43.758}$	$1.06\overline{)\,31.472}$

make good the discrepancy. Pupils should understand and use the rule given; they should not be asked to recite it verbatim.

Devices for locating the decimal point. Various devices for locating the decimal point in the quotient are used. There are two which, with minor variations, are most common. They are:

1. Move your pencil to the right of the decimal point in the dividend as many places as there are decimal places in the divisor. Now put the decimal point in the quotient directly above this position. Annex zeros if necessary.

2. Multiply the divisor by 10, 100, 1000, or whatever is required to change it to a whole number. Multiply the dividend by the same number and divide.

The dispute as to which is the better device usually waxes strongest among those who supply pupils with a ready-made rule for pointing off the quotient and who supply no explanation of the origin of the rule. No device should be used until pupils have learned why the point is placed where it is and until they have practised enough to be able to locate the point with assurance without the aid of a device. The former device has an advantage in that the numbers used as divisor and dividend are not changed from their original form and in that it is, after all, but a variation of the principle that the number of decimal places in the divisor plus the number in the quotient must equal the number in the dividend. The latter has a slight advantage in that it makes all examples uniform to the extent that the divisor is always an integer. If the pupils understand and use the principle that *Divisor* $\times$ *Quotient* = *Divi-*

dend, if they check their solutions by multiplication, and if they inspect each quotient to see that it is a sensible one, they are likely to obtain correct results. And if the results are correct and the procedure is understood, it matters little what trick or device is used to enable one to tell where the point goes.

Tests in multiplication and division with decimals. Long examples in multiplication or division with decimals when used as tests fail in their purpose because they involve much multiplication or division and few decimals. The example shown, for instance, may require minutes for the operation of division but only seconds for locating the decimal point in the quotient. Manifestly, if we wish to test the pupil's ability to solve examples in division with decimals we obscure the issue by requiring from him an exorbitant amount of long division.

```
          17.9
4.21) 75.632
        421
        3353
        2947
         4062
         3789
          273
```

For this reason, it seems best to follow Monroe's suggestion for testing pupils' ability in multiplication and division with decimals. He supplies the products and quotients on a printed test paper and simply requires pupils to insert the decimal points in the proper places.[4]

A caution is in order in connection with the use of the Monroe tests or similar tests, however. The time allowance is always very brief on account of the simple nature of the task required. The first time a pupil takes such a test, he will probably consume a considerable proportion of his brief time allowance getting oriented,

[4] Monroe, Walter Scott. *Measuring the Results of Teaching.* Boston: Houghton Mifflin Company, 1918, pp. 115-116.

getting started on the items of the test. The next time he takes the test, however, he knows better what he is to do, starts promptly, and usually makes a much better score. The gain is clearly more apparent than real. The author has found that the first application of such a test is likely to yield a very inadequate measure of the pupil's ability in multiplication or division with decimals, especially if only a small amount of time is allowed. It seems to be better to increase considerably the amount of time allowed on such tests for speed in locating decimal points is not very important.

QUESTIONS AND REVIEW EXERCISES

1. Why are decimal fractions called "decimal" fractions? If our number system were a duodecimal system what would the number 4.1 mean?

2. Some persons recommend that very little attention be given to common fractions except to change them to decimals. Do you believe that pupils who have had very little experience with common fractions would, for this reason, find decimals more difficult?

3. Why does it seem better to make the first approach to the study of decimal fractions through the medium of common fractions instead of through the pupil's understanding of our system of notation? Should both of these approaches be used?

4. Are integers decimal numbers?

5. When the approach to decimals by way of our system of notation has been emphasized, pupils sometimes read such a decimal as .428 as "four hundred twenty-eight hundredths." Do you see any reasonable explanation for their saying "hundredths" for a three-place decimal?

6. Should pupils eventually memorize the fact that one decimal place means tenths, that two decimal places mean

hundredths, and that three decimal places mean thousanths? Why?

7. Which is more important, the ability to read decimals or the ability to write them from dictation? Why?

8. Where should the word "and" be used in reading numbers? Are there any exceptions?

9. Is it better for a pupil to read .305 as "point, three, zero, five" or "point, three, O, five?" Why?

10. Do pupils in the primary grades ordinarily have any experience in reading decimal mixed numbers?

11. What new things does a pupil have to learn about the function of zeros when he learns decimal fractions?

12. Is there any difference between the meaning of .67 and .670? If so, what is the difference?

13. Under what circumstances would you recommend that common fractions be converted to decimal fractions and under what circumstances would you have them left in common fraction form?

14. How would you teach pupils to change common fractions to decimal fractions?

15. Do you believe that the average individual uses decimal fractions as much as he should? How may a teacher encourage a greater use of decimals?

16. What common fractions would you include in a list for which pupils are to memorize decimal equivalents? How soon after they begin the study of decimals would you have the pupils memorize these equivalents?

17. According to Brueckner, what are the commonest errors which pupils make in decimal fractions?

18. What is there new or difficult for the pupils in learning to add decimals? Why do you think pupils sometimes obtain .13 as the sum of .4, .4, and .5?

19. Where does the greatest difficulty lie in learning subtraction of decimals?

20. What advantage may pupils gain from exercises in arranging series of decimal fractions in order of size?

21. How would you rationalize the fact that if a decimal is multiplied by an integer there are as many decimal places in the product as in the multiplicand?

22. How would you rationalize the fact that if an integer is multiplied by a decimal there are as many decimal places in the product as in the multiplier? Do you see two ways to rationalize this?

23. Summarize the advantages in having pupils study the similarity between multiplication of decimal fractions and multiplication of common fractions.

24. Why does it seem best to begin the study of division with decimals with examples in which decimals are to be divided by integers?

25. How extensively would you have pupils study the cycles of digits yielded by circulating decimals?

26. What advantage is there in stressing the measurement idea of division in the division of integers by decimals?

27. What is the check of common sense in division with decimals? Can this check be used in multiplication with decimals also?

28. How would you teach pupils where the point in the quotient should be placed when decimals are divided by decimals?

29. What devices do you know for placing the decimal point in the quotient? What use would you make of these devices?

30. Would you use Monroe's plan in constructing tests in multiplication and division with decimals? What danger is there in the use of this kind of test?

CHAPTER TEST

For each of these items select the best answer. A scoring key will be found on page 534.

1. A decimal fraction is one whose denominator is (1) 10 (2) a power of 10 (3) a multiple of 10.

2. The word "and" should be used to separate the whole number from the fraction when reading (1) common fractions (2) decimal fractions (3) both common and decimal fractions.

3. Moving the decimal point one place to the right (1) multiplies the number by 10 (2) divides the number by 10 (3) increases the number by 10.

4. The number .4603 is read (1) four thousand six hundred three ten thousandths (2) four thousand six hundred three tens thousandths (3) four thousand six hundred and three ten thousandths.

5. In studying decimal fractions, the pupils should use (1) the approach through common fractions only (2) the approach through our decimal system of notation only (3) both the approach through common fractions and the approach through the system of notation.

6. The ability to write decimal fractions from dictation compared with the ability to read them is (1) more important (2) less important (3) equally important.

7. If $1/15$ is changed to a three-place decimal, the correct answer is (1) .667 (2) .666 (3) .067.

8. Brueckner found the largest number of errors in decimals in (1) addition (2) multiplication (3) division.

9. Brueckner found the fewest errors in (1) subtraction (2) multiplication (3) division.

10. If a pupil obtains .15 as the sum of .6, .7, and .2, the teacher should (1) correct his mistake (2) have him check his answer by using common fractions (3) tell him

that the number of decimal places in the sum can not be greater than the greatest number in the addends.

11. The rule for pointing off the product in multiplication with decimals (1) should be memorized by the pupils before they solve any examples (2) should be discovered by pupils themselves (3) should be stated by the teacher after illustrative examples have been solved.

12. To show the product of tenths by tenths, the first illustrative example may well be (1) $\begin{array}{r}.9\\ \underline{.4}\end{array}$ (2) $\begin{array}{r}.3\\ \underline{.2}\end{array}$ (3) $\begin{array}{r}4.3\\ \underline{5.7}\end{array}$

13. If the multiplication of thousandths by thousandths is to be illustrated with common fractions, the number of illustrative examples should be (1) 1 (2) 3 (3) 6.

14. The first work in division with decimals should be (1) division of a decimal by an integer (2) division of an integer by a decimal (3) division of a decimal by a decimal.

15. When there is a remainder in division and the quotient is carried to two decimal places, the quotient should be expressed to the (1) next lower hundredth (2) nearest hundredth (3) next higher hundredth.

16. Errors in locating the decimal point in the quotient are best detected by (1) repeating the work by the method of common fractions (2) inspecting the answer to see if it is a reasonable one (3) employing multiplication as a check.

Read each statement and decide whether it is true or false. The scoring key will be found on page 534.

17. Every common fraction can be expressed exactly as a decimal fraction.

18. Every decimal fraction can be expressed exactly as a common fraction.

19. It was recommended that the first approach to the study of decimal fractions be through related common fractions.

20. Reading numbers representing amounts of money supplies most pupils with their first experiences in reading decimal mixed numbers.

21. Arithmetically, .4 = .40.

22. If a carefully made measurement is reported to be 0.016 in., the true measurement may be as much as 0.0164 in.

23. The typical individual uses decimal fractions more than he should.

24. The recommended list of common fractions whose decimal equivalents are to be memorized included the fraction, $\frac{1}{12}$.

25. Pupils eventually should memorize the fact that three decimal places mean thousandths.

26. If a decimal is multiplied by the nth power of 10, the decimal point is moved n places to the right.

27. The number of major classes of examples recognized in division with decimals is greater than the number of classes recognized in multiplication.

28. If a decimal is divided by an integer and the quotient figures are correctly placed, the decimal point in the quotient will be directly above the decimal point in the dividend.

29. Circulating decimals will come out even if the division is carried far enough.

30. All pupils should study extensively the subject of circulating decimals.

31. In teaching the division of an integer by a decimal, the partition idea of division should be emphasized.

32. Devices for locating the decimal point should be introduced early in the pupil's experience in division with decimals.

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11. Westaway, F. W. *Craftsmanship in the Teaching of Elementary Mathematics.* London: Blackie and Son, Limited, 1931, 665 pp. Chapter X, pages 78-94, presents a brief but well written treatment of decimal fractions.

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CHAPTER 8

THE ELEMENTS OF PERCENTAGE

A story is told to the effect that one of the state legislatures was debating a bill which provided that county school bonds could not be issued unless 80 per cent of the votes were in favor. One member shouted, "Mr. Speaker, I protest. This bill will cut out my county entirely for there ain't that many people in the county."

A line coach at one of the large state universities was discussing the troubles of a coach in building a good football team. "Last spring," he said, "we had 23 men out for spring practice. This fall only eight of them came back to school. That is a loss of 15 per cent and you can't build a football team that way."

These incidents suggest that persons in high places may have a very inadequate understanding of percentage. There are those who conclude that percentage must be difficult since so many persons do not understand it. But if it is well taught, percentage is not difficult.

A new term for a familiar idea. The subject matter of percentage is neither new nor difficult for those who know and understand common and decimal fractions. Familiar numbers which have been known as fractions, usually as decimal fractions, are dressed up in new clothes, as it were. *Per cent* means *hundredths*. Thus,

$$.14 = 14 \text{ hundredths} = 14 \text{ per cent}$$
$$.62 = 62 \text{ hundredths} = 62 \text{ per cent}$$
$$.06 = 6 \text{ hundredths} = 6 \text{ per cent}$$

In spite of the fact that percentage is not new and should not be difficult, it is often observed that pupils do have difficulty with the subject. There seem to be two chief reasons for this.

First, pupils undertake to study percentage without an adequate understanding of the underlying fraction ideas. Pupils who do not understand common fractions well move on to the study of decimal fractions. They learn a little of decimals, largely by rote, and then try percentage. Naturally, percentage is meaningless to them. We have here a striking illustration of the disastrous effects of a method of instruction which is based upon the drill theory. If teaching has proceeded according to the standards of the meaning theory, percentage should be relatively easy.

Secondly, percentage is sometimes taught in such a way as to increase instead of diminish the difficulty of the subject for the pupils. Teachers make their initial explanations as if the idea involved were entirely new to the pupils. Instead of stressing the fact that percentage is merely a new word for an idea with which the pupils are already familiar, they present both the word and the idea as if they were new. Furthermore, early efforts at formal definitions becloud the issue especially when an effort is made to define *per cent* in terms of its Latin origin. Textbooks also contribute to the difficulty. One textbook begins the explanation of the subject as follows:

> Per cent means "by the hundred." The expression is an abbreviation of the Latin *per centum*.

This explanation is not only hard for children to understand but also it is actually misleading. Per cent does

not mean "by the hundred" whatever the literal meaning of the Latin words from which the expression is derived. Furthermore, children in the sixth or the seventh grade neither know nor care about the Latin origin of our English words. Such etymological excursions may be both interesting and illuminating to the teacher (if the teacher has studied Latin) but they have no place in the training of the pupil who has not studied Latin. They are not likely to add either to his understanding of the subject or to his interest in it.

It is important that the meaning of "per cent" be made clear to the pupils at the very beginning of their study of the subject. To make it clear we need merely to inform pupils that business men and others often use the term "per cent" when they refer to hundredths. Thus, when the shoe dealer says that he sold 50 per cent of his stock of shoes during his sale, he means that he sold 50 hundredths, or .50, or ½ of his stock; when a merchant advertises a reduction of 25 per cent on girls' coats, he means that he has reduced the price 25 hundredths, or .25, or ¼; when a man sells a building lot at a profit of 20 per cent, he makes a profit of 20 hundredths, or .20, or ⅕. "Per cent" means "hundredths"; "per cent of" means "hundredths of"; "per cent times" means " hundredths times."

If the first lessons in percentage are planned so as to show the close relationship between percentage and decimal fractions or common fractions, pupils will make the transition from fractions to percentage easily and quickly. On the other hand, if percentage is presented as a *new* subject with little or no effort to show the

intimate relationship existing between it and decimal fractions (and, less directly, between it and common fractions) pupils may be expected to master it slowly and with difficulty, if they master it at all. Realizing that percentage is an extremely important subject and that it is one which finds many applications in social and business affairs, many teachers make their first lessons in this subject both formal and formidable. It is much better to plan the first lessons so as to encourage the pupils to believe that they are merely learning a new way to express that which they already know.

The four fundamental processes. There are four fundamental processes in percentage with which pupils should become well acquainted. These processes are used when percentage is applied to problem situations. The processes are:

1. Changing decimal fractions to per cents
2. Changing per cents to decimal fractions
3. Changing common fractions to per cents
4. Changing per cents to common fractions

Changing decimal fractions to per cents. First, pupils should have practice in changing decimal fractions to per cents. This should not be difficult if *per cent* has been defined as merely another word for *hundredths*. Pupils should be taught first to express decimal fractions as hundredths and then as per cents. The symbol, %, should be introduced at this time. The first exercises should be easy cases of two-place decimals. After writing on the blackboard two or three illustrative examples in the form shown on page 370, the teacher may assign

other examples for the pupils to write out in the same manner.

.15 = 15 hundredths = 15 per cent = 15%
.75 = 75 hundredths = 75 per cent = 75%
.06 = 6 hundredths = 6 per cent = 6%

One-place decimals may be used next. The following examples are illustrative.

.5 = .50 = 50 hundredths = 50 per cent = 50%
.8 = .80 = 80 hundredths = 80 per cent = 80%
.1 = .10 = 10 hundredths = 10 per cent = 10%

Next, the more troublesome case of three-place decimals may be introduced. Pupils have already discovered that the words "per cent," or the symbol "%," take the place of two decimal places. Then, the teacher may show that if the decimal has three places and the per cent sign is to take the place of two of these three places, one decimal place will remain. In other words, a three-place decimal may be expressed as hundredths by moving the point two places to the right. Thus,

.425 = 42.5 hundredths = 42.5 per cent = 42.5%
.625 = 62.5 hundredths = 62.5 per cent = 62.5%
.375 = 37.5 hundredths = 37.5 per cent = 37.5%
.042 = 4.2 hundredths = 4.2 per cent = 4.2%
.018 = 1.8 hundredths = 1.8 per cent = 1.8%
.004 = .4 hundredths = .4 per cent = .4%

Pupils will see that such an expression as .4% means .4 of one per cent.

As pupils practise in the writing of these per cent

equivalents, they will learn gradually to abbreviate the form of expression which we have been using, eliminating first one step and then two steps. Thus, they will write, $.425 = 42.5$ hundredths $= 42.5\%$ and finally, $.425 = 42.5\%$. Then, miscellaneous decimal fractions will be written directly as per cents, as follows:

$.45 = 45\%$	$.50 = 50\%$	$.125 = 12.5\%$
$.6 = 60\%$	$.05 = 5\%$	$.083 = 8.3\%$
$.03 = 3\%$	$.37 = 37\%$	$.005 = .5\%$

Four-place decimals may be used also but these are of less practical value and should be considered later.

Changing per cents to decimal fractions. When pupils have become proficient in changing decimal fractions to per cents they should not find it difficult to learn to reverse the process and change per cents to decimal fractions. The following examples will illustrate some of the various types which should be included in sets of practice exercises.

$25\% = .25$	$37.5\% = .375$
$16\% = .16$	$28.3\% = .283$
$5\% = .05$	$4.5\% = .045$
$82\% = .82$	$5.3\% = .053$

Pupils will remember that per cent means hundredths and will see, then, that if the per cent sign is removed, two decimal places must be pointed off. If the per cent expression already has a decimal point, the removal of the per cent sign means that this point must be moved two places to the left.

Per cents expressed in the form of common fraction

mixed numbers may be introduced at this point. As the per cents are changed to decimals the common fractions may be changed to the decimal form as illustrated by the following examples.

$$5\tfrac{1}{2}\% = .05\tfrac{1}{2} = .055$$
$$37\tfrac{1}{2}\% = .37\tfrac{1}{2} = .375$$
$$4\tfrac{1}{4}\% = .04\tfrac{1}{4} = .0425$$
$$3\tfrac{3}{4}\% = .03\tfrac{3}{4} = .0375$$

In the event that a fraction yields a circulating or other non-terminating decimal, pupils will follow their usual practice of expressing the decimal to the nearest thousandth, etc., and of discarding the fractional remainder. Thus,

$$33\tfrac{1}{3}\% = .33\tfrac{1}{3} = .3333$$
$$16\tfrac{2}{3}\% = .16\tfrac{2}{3} = .1667$$
$$66\tfrac{2}{3}\% = .66\tfrac{2}{3} = .6667$$

Changing common fractions to per cents. In changing common fractions to per cents, pupil will first convert them to two-place decimals and then to per cents. In the preceding chapter we have discussed at some length the conversion of common fractions to decimal fractions but have indicated that pupils will express their results as tenths, thousandths, etc., as well as hundredths. Then, the only novel phase of the present situation is the fact that the common fractions are to be expressed as hundredths in all cases since per cent means hundredths.

Several illustrative examples should be placed upon the blackboard and explained. The following are typical.

$\frac{1}{2} = .50 = 50\%$	$\frac{1}{3} = .33\frac{1}{3} = 33\frac{1}{3}\%$
$\frac{1}{4} = .25 = 25\%$	$\frac{1}{6} = .16\frac{2}{3} = 16\frac{2}{3}\%$
$\frac{3}{4} = .75 = 75\%$	$\frac{5}{6} = .83\frac{1}{3} = 83\frac{1}{3}\%$
$\frac{1}{5} = .20 = 20\%$	$\frac{2}{3} = .66\frac{2}{3} = 66\frac{2}{3}\%$
$\frac{2}{5} = .40 = 40\%$	$\frac{1}{8} = .12\frac{1}{2} = 12\frac{1}{2}\%$
$\frac{1}{10} = .10 = 10\%$	$\frac{1}{12} = .08\frac{1}{3} = 8\frac{1}{3}\%$

There are certain points which should be emphasized in changing common fractions to per cents. In the first place, it is sometimes necessary to write a 0 as the last decimal figure. Thus, $\frac{1}{2} = .50 = 50\%$. Heretofore, pupils have been accustomed to writing $\frac{1}{2} = .5$ but not $\frac{1}{2} = .50$. But per cent means hundredths, so we must always have two decimal places.

In the second place, in changing common fractions to hundredths we frequently have a fractional remainder left after two decimal places have been written. Thus, $\frac{1}{8} = .12\frac{1}{2} = 12\frac{1}{2}\%$. The pupils have been accustomed to writing $\frac{1}{8} = .125$ but not $\frac{1}{8} = .12\frac{1}{2}$. Again, a pupil must remember that he should always have two decimal places. Of course, he can write $\frac{1}{8} = .125 = 12.5\%$ but $12\frac{1}{2}\%$ is a more common form of expression than 12.5%. However, a pupil should be made acquainted with both of these forms for both are used.

In the preceding chapter, we recommended that pupils memorize the decimal equivalents of certain common fractions of frequent occurrence. This list should be reviewed now and the per cent equivalent of each of the common fractions should be well fixed in memory. When expressed as per cents or as cents these fractions are frequently called *aliquot parts*. The minimum list to be memorized follows. We give for each

fraction not only the per cent equivalent but also the number of cents in that fractional part of a dollar.

$\frac{1}{2} = 50\%$	$\frac{1}{2}$ of a dollar = 50 cents
$\frac{1}{3} = 33\frac{1}{3}\%$	$\frac{1}{3}$ of a dollar = $33\frac{1}{3}$ cents
$\frac{2}{3} = 66\frac{2}{3}\%$	$\frac{2}{3}$ of a dollar = $66\frac{2}{3}$ cents
$\frac{1}{4} = 25\%$	$\frac{1}{4}$ of a dollar = 25 cents
$\frac{3}{4} = 75\%$	$\frac{3}{4}$ of a dollar = 75 cents
$\frac{1}{5} = 20\%$	$\frac{1}{5}$ of a dollar = 20 cents
$\frac{2}{5} = 40\%$	$\frac{2}{5}$ of a dollar = 40 cents
$\frac{3}{5} = 60\%$	$\frac{3}{5}$ of a dollar = 60 cents
$\frac{4}{5} = 80\%$	$\frac{4}{5}$ of a dollar = 80 cents
$\frac{1}{6} = 16\frac{2}{3}\%$	$\frac{1}{6}$ of a dollar = $16\frac{2}{3}$ cents
$\frac{5}{6} = 83\frac{1}{3}\%$	$\frac{5}{6}$ of a dollar = $83\frac{1}{3}$ cents
$\frac{1}{8} = 12\frac{1}{2}\%$	$\frac{1}{8}$ of a dollar = $12\frac{1}{2}$ cents
$\frac{3}{8} = 37\frac{1}{2}\%$	$\frac{3}{8}$ of a dollar = $37\frac{1}{2}$ cents
$\frac{5}{8} = 62\frac{1}{2}\%$	$\frac{5}{8}$ of a dollar = $62\frac{1}{2}$ cents
$\frac{7}{8} = 87\frac{1}{2}\%$	$\frac{7}{8}$ of a dollar = $87\frac{1}{2}$ cents
$\frac{1}{10} = 10\%$	$\frac{1}{10}$ of a dollar = 10 cents
$\frac{3}{10} = 30\%$	$\frac{3}{10}$ of a dollar = 30 cents
$\frac{7}{10} = 70\%$	$\frac{7}{10}$ of a dollar = 70 cents
$\frac{9}{10} = 90\%$	$\frac{9}{10}$ of a dollar = 90 cents

While these aliquot parts of one hundred per cent or of a dollar are receiving practice it is well to review certain fractions which have been omitted from this list because, when reduced to lowest terms, each becomes one of the fractions which has been included. These fractions, with their equivalents, are:

$\frac{2}{4} = \frac{1}{2}$	$\frac{2}{8} = \frac{1}{4}$	$\frac{4}{10} = \frac{2}{5}$
$\frac{2}{6} = \frac{1}{3}$	$\frac{4}{8} = \frac{1}{2}$	$\frac{5}{10} = \frac{1}{2}$
$\frac{3}{6} = \frac{1}{2}$	$\frac{6}{8} = \frac{3}{4}$	$\frac{6}{10} = \frac{3}{5}$
$\frac{4}{6} = \frac{2}{3}$	$\frac{2}{10} = \frac{1}{5}$	$\frac{8}{10} = \frac{4}{5}$

A valuable form of oral practice consists of having the pupils count to 100 per cent (or to 1) by using these fractional parts, such as sixths or eighths. Thus, if a pupil counts to 100 per cent by sixths, he will say, $16\frac{2}{3}$ per cent, $33\frac{1}{3}$ per cent, 50 per cent, $66\frac{2}{3}$ per cent, $83\frac{1}{3}$ per cent, 100 per cent. Using eighths, he will say, $12\frac{1}{2}$ per cent, 25 per cent, $37\frac{1}{2}$ per cent, 50 per cent, $62\frac{1}{2}$ per cent, 75 per cent, $87\frac{1}{2}$ per cent, 100 per cent. This exercise is difficult at first but when practised it helps pupils to remember the more difficult of these aliquot parts. They should begin with fourths, fifths, and tenths since these are very much easier.

Changing per cents to common fractions. The fourth and last of the fundamental processes of percentage with which pupils should be made acquainted in order that they may know and understand percentage better is changing per cents to common fractions. Since this is simply the reverse of the procedure discussed in the preceding section, it should not be difficult for pupils to learn and understand. They first write the decimal equivalent of the percentage form, then change this decimal fraction to a common fraction and, finally, reduce the common fraction to lowest terms. A few blackboard illustrations will make clear what is to be done.

$$
\begin{aligned}
50\% &= .50 = \tfrac{50}{100} = \tfrac{1}{2}\\
25\% &= .25 = \tfrac{25}{100} = \tfrac{1}{4}\\
40\% &= .40 = \tfrac{40}{100} = \tfrac{2}{5}\\
4\% &= .04 = \tfrac{4}{100} = \tfrac{1}{25}\\
1\% &= .01 = \tfrac{1}{100}\\
15\% &= .15 = \tfrac{15}{100} = \tfrac{3}{20}
\end{aligned}
$$

$$64\% = .64 = {}^{64}/_{100} = {}^{16}/_{25}$$
$$78\% = .78 = {}^{78}/_{100} = {}^{39}/_{50}$$
$$31\% = .31 = {}^{31}/_{100}$$

Pupils should be provided with several examples for practice. Examples like these not only help pupils to understand percentage but also they provide a valuable review of previous work in changing decimal fractions to common fractions and in reducing common fractions to lowest terms.

All of the fractions used in the list of aliquot parts should be included in the list of examples given for practice in changing per cents to common fractions. These will include the more common fractional forms, such as $33\frac{1}{3}\%$, $66\frac{2}{3}\%$, $16\frac{2}{3}\%$, $83\frac{1}{3}\%$, $12\frac{1}{2}\%$, $37\frac{1}{2}\%$, etc. The pupil will write,

$$12\tfrac{1}{2}\% = .12\tfrac{1}{2} = .125 = {}^{125}/_{1000} = \tfrac{1}{8},$$
$$37\tfrac{1}{2}\% = .37\tfrac{1}{2} = .375 = {}^{375}/_{1000} = \tfrac{3}{8}, \text{ etc.}$$

The forms, $33\frac{1}{3}\%$, $66\frac{2}{3}\%$, $16\frac{2}{3}\%$, and $83\frac{1}{3}\%$ do not lend themselves so readily to such treatment and are considerably more difficult. Of course, we can write,

$$33\tfrac{1}{3}\% = .33\tfrac{1}{3} = \frac{33\frac{1}{3}}{100} = \frac{{}^{100}/_{3}}{100} = {}^{100}/_{300} = \tfrac{1}{3},$$

but these complex fraction forms are more unusual and should come later. Since the pupil should know the fraction equivalents of these per cents by this time, he should write simply,

$$33\tfrac{1}{3}\% = .33\tfrac{1}{3} = \tfrac{1}{3}$$
$$66\tfrac{2}{3}\% = .66\tfrac{2}{3} = \tfrac{2}{3}$$
$$16\tfrac{2}{3}\% = .16\tfrac{2}{3} = \tfrac{1}{6}$$
$$83\tfrac{1}{3}\% = .83\tfrac{1}{3} = \tfrac{5}{6}$$

The more unusual forms, such as $55\frac{5}{9}\%$, $85\frac{5}{7}\%$, etc., —forms which belong to the non-terminating decimal type and which have not been included in the list of aliquot parts—should not be given for conversion to common fractions until the pupils have had considerable practice with complex fractions, if they are given at all. (These per cents are equal to $\frac{5}{9}$ and $\frac{6}{7}$, respectively.)

After several days spent in practice on these four fundamental processes, pupils should have sufficient understanding of the meaning of percentage to enable them to take up the various cases and to solve the simpler problems involving the use of percentage.

Per cents greater than one hundred. Pupils frequently have difficulty with per cents greater than 100. Ordinarily, most of their practice is with per cents less than 100, frequently less than 50, and, as a consequence, they have no ready means of interpreting per cents greater than 100 when they are encountered. They are frequently told that 100 per cent of anything is all of it; naturally, then, more than 100% is not meaningful.[1]

Per cents over 100 may be used with each of the four fundamental processes which have been discussed in the preceding pages. The following examples will suggest the kinds which may be used for illustrations and for practice.

1.75 = 175 hundredths = 175 per cent = 175%
2.5 = 250 hundredths = 250 per cent = 250%
4 = 400 hundredths = 400 per cent = 400%
1 = 100 hundredths = 100 per cent = 100%

[1] Cf. Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, p. 155.

$$\begin{aligned}
115\% &= 115 \text{ hundredths} = 1.15 \\
150\% &= 150 \text{ hundredths} = 1.50 = 1.5 \\
200\% &= 200 \text{ hundredths} = 2.00 = 2 \\
500\% &= 500 \text{ hundredths} = 5.00 = 5
\end{aligned}$$

$$\begin{aligned}
5/4 = 1\tfrac{1}{4} = 1.25 &= 125 \text{ hundredths} = 125\% \\
8/5 = 1\tfrac{3}{5} = 1.6 &= 160 \text{ hundredths} = 160\% \\
2\tfrac{1}{2} = 2.5 &= 250 \text{ hundredths} = 250\% \\
1\tfrac{3}{8} = 1.375 &= 137.5 \text{ hundredths} = 137.5\%
\end{aligned}$$

$$\begin{aligned}
125\% &= 125 \text{ hundredths} = 1.25 = 1\tfrac{1}{4} \\
250\% &= 250 \text{ hundredths} = 2.5 = 2\tfrac{1}{2} \\
133\tfrac{1}{3}\% &= 133\tfrac{1}{3} \text{ hundredths} = 1.33\tfrac{1}{3} = 1\tfrac{1}{3} \\
187\tfrac{1}{2}\% &= 187\tfrac{1}{2} \text{ hundredths} = 1.875 = 1\tfrac{7}{8}
\end{aligned}$$

Each of these four type forms may be written out in full as given above until the pupils understand well what the expressions mean; then they may be expressed more briefly.

To add further to the pupils' understanding of per cents greater than 100, realistic problems in which such per cents arise should be solved. For example, certain rapidly growing cities which more than double their populations in a given period of time yield per cent increases in population greater than 100. A building lot in a rapidly expanding community is worth today three times as much as it was 10 years ago and, therefore, has increased in value 200 per cent. If John weighs twice as much as his younger brother, he weighs 100 per cent more than his younger brother. Other like illustrations can be thought of.

The three type problems of percentage. There are three types of percentage problems. Usually, these are

called, Case I, Case II, and Case III. The three types may be illustrated by examples.

1. What is 6% of $200? (Case I)
2. $12 is what per cent of $200? (Case II)
3. $12 is 6% of what amount? (Case III)

It will be seen that three items are involved in these examples. They are: (1) the base, here $200; (2) the rate, here 6%; and (3) the percentage, here $12. In each case, two of these three are given and the third is to be found. When the percentage is to be found, Case I is used; when the rate is to be found, Case II is used; and when the base is to be found, Case III is used.

The first of these three types occurs very frequently in the common applications of percentage as will be seen in later pages. The second type is found rather often in business affairs but not so frequently as the first type. The third type is of relatively rare occurrence but is sometimes met in certain special forms of problems in the uses of percentage in business, especially in problems which have to do with the calculation of profits on the basis of the selling price. All three types should be included in the course of study since each supplements the others in developing the pupil's understanding of percentage. The third type should be given relatively less emphasis than the second and the second less than the first.

Teaching the first type. In solving problems of the first type, one simply multiplies by a fraction, decimal or common. A problem may be stated and the solution developed as follows:

TEACHER: "A man, whose salary is $3600 a year, decides that he will try to save 25% of it. How much does he plan to save? 25% of his salary is what *part* of his salary?"

PUPIL: "One-fourth."

TEACHER: "Then, how much does he plan to save?"

PUPIL: "$900."

TEACHER: "Suppose that he planned to save 15% of his salary. Since per cent means hundredths, he plans to save .15 of his salary. How will you find the amount he plans to save?"

PUPIL: "Find .15 of $3600."

TEACHER: "How do you find .15 of $3600?"

PUPIL: "Multiply $3600 by .15."

TEACHER: "Now use the same method to find 25% of his salary. Of course, this also gives $900 since .25 is the same as ¼ and both are the same as 25%."

$$\begin{array}{r} \$3600 \\ .15 \\ \hline 18000 \\ 3600 \\ \hline \$540.00 \end{array}$$

Other illustrative examples and problems should be solved on the blackboard. These should include per cents which are not aliquot parts of 100 as well as those which are. Pupils should be encouraged to look for the easiest and shortest solution. If the per cent is one of the aliquot parts which have been learned, the common fraction equivalent should be used as multiplier; otherwise, the multiplier will be a decimal fraction. Problems and examples for practice are easily prepared.

In preparing problems for practice on Case I, it is well to anticipate some of the easier applications of percentage which will be studied more intensively later. The following problems suggest the kinds which may be used.

1. Robert sold 24 tickets to the school play at 35 cents each. He was allowed to keep 10% of the money for selling the tickets. How much did he keep?

2. Clara's grandfather gave her $10 for Christmas. She put the money in a savings bank. The bank paid her 3% interest at the end of the year. How much interest did the bank pay Clara?

3. John's father buys and sells second-hand automobiles. One morning he bought a car for $250. The next day he sold the car and said, "I made a profit of 30% on that car." How much did he make?

4. Each property owner in our city pays 2% of the value of his property as taxes. How much tax does Mr. Richards pay on a house worth $5500?

The first problem deals with commission, the second with interest, the third with profit and loss, and the fourth with taxes. Pupils enjoy these little excursions into the arithmetic of business if the problems are simply worded and if they deal with familiar situations. It may be necessary to explain some of the terms used, such as *interest, profit,* and *taxes.*

Teaching the second type. If a man buys a house for $6000 and sells it for $6500, he knows readily enough that his profit is $500 but he is likely to want to know the per cent gained on his investment also. In other words, he wants to know what per cent $500 is of $6000. This type of problem occurs rather frequently in the affairs of some persons and requires for its solution the second case of percentage. We may develop this type of percentage solution in class by using a problem which is simpler and more meaningful to the pupils.

TEACHER: "There are 30 pupils in our room, 18 boys and 12 girls. What per cent are boys? What per cent are girls? Remember that per cent means hundredths. We must find how many *hundredths* of the pupils are boys. Let us first see what *part* of the pupils are boys. If there are 18 boys and 30 pupils altogether, what part of the pupils are boys?"

PUPIL: "Eighteen-thirtieths."

TEACHER: "Reduce $\frac{18}{30}$ to lowest terms. What do you get?"

PUPIL: "Three-fifths."

TEACHER: "Then, $\frac{3}{5}$ of the pupils are boys; $\frac{3}{5}$ is the same as what per cent?"

PUPIL: "Sixty per cent."

TEACHER: "You see, we first find what *part* of the class are boys. Then we change that fraction to per cent. Now, see if you can find what per cent of the class are girls."

PUPIL: "Twelve-thirtieths are girls. But $\frac{12}{30} = \frac{2}{5} =$ 40%. So, 40% are girls."

TEACHER: "We see that 60% of the class are boys and 40% are girls. If we add the per cent of boys and the per cent of girls, we have $60\% + 40\% = 100\%$. But 100% of the class is all of the class, so our two answers are probably correct."

As another check, the pupils may find 60% of 30 and 40% of 30, thus reviewing Case I.

The class may find the per cent of boys and girls in each room in the building, or in each grade in the room if more than one grade is instructed by the teacher. This will afford excellent practice in changing various unusual common fractions to hundredths and to per cents. Suppose that an elementary school has six rooms, one for each of the grades one to six, inclusive, and that the

enrollments are those given in Table 6. The pupils may find the per cent of boys and girls in each room and in the entire building and set down the per cents in the proper spaces in the table. For the teacher's convenience, the per cents have been calculated and are recorded in the table.

TABLE 6. ENROLLMENTS OF BOYS AND GIRLS IN AN ELEMENTARY SCHOOL, WITH PER CENTS

Grade	Enrollment			Per Cent		
	Boys	Girls	Total	Boys	Girls	Total
I	25	20	45	56	44	100
II	14	18	32	44	56	100
III	18	17	35	51	49	100
IV	20	18	38	53	47	100
V	17	19	36	47	53	100
VI	14	16	30	47	53	100
Total	108	108	216	50	50	100

In changing a common fraction to a per cent pupils should be taught to express the quotient to the *nearest* hundredth in each case just as they have been taught to express the quotient to the nearest tenth, hundredth, thousandth, etc., in changing a common fraction to a decimal. Failure to do this results frequently in per cents less than 100 in the total column.

Later, per cents may sometimes be expressed in three figures with a decimal point after the second figure. Suppose that there are 33 pupils in a room and that 16 of them are boys and 17 are girls. The results of the pupil's work would appear as follows:

Enrollment			Per Cent		
Boys	Girls	Total	Boys	Girls	Total
16	17	33	48.5	51.5	100.0

The following problems are suggestive of the kinds which may be used in practising on percentage problems which belong to Case II. They have been drawn from a wide variety of sources and are intended merely to serve as suggestions to the teacher.

1. The Dickson family drove to Chicago, 475 miles away. The first day they drove 285 miles. What per cent of the distance had they traveled?

2. Dick bought a pair of skates which had been marked down from $4.00 to $3.25. What per cent of the $4.00 did he save?

3. The Jacksons have 12 rows of potatoes growing in their garden. One Saturday Ralph and his brother Charles hoed all 12 rows. Ralph hoed 7 rows and Charles hoed 5. What per cent of the rows did Ralph hoe? What per cent did Charles hoe?

4. Last week Elizabeth earned $1.75 taking care of Mrs. Smith's baby. On Saturday, she spent 75 cents for a pair of gloves. What per cent of her money did she spend?

5. Last year the Pattersons burned 11 tons of coal in their furnace. This year they burned only 9 tons. What per cent was this year's coal of last year's?

6. James is saving his money for a pair of skates which cost $6.50. He has saved $4.63. What per cent of the $6.50 has he saved?

7. Harry weighs 82 pounds and John weighs 65 pounds. John's weight is what per cent of Harry's? Harry's weight is what per cent of John's?

8. Frank sells papers every evening. Last evening he got 50 papers and sold 43 of them before supper. What per cent of the papers did he sell before supper?

9. The Beaver Ridge High School football team won five games last season and lost four. Find to the nearest tenth of one per cent the per cent of games won.

10. Mr. Jordan has two cars. He gets 13 miles from a gallon of gasoline with the big car and 17 miles from a gallon with the little car. What per cent of 17 is 13? What per cent of 13 is 17?

It will be seen that problems 7 and 10 in this list require pupils to find per cents which are greater than 100. This kind of percentage problem is frequently neglected in textbooks. It affords an interesting variation in the work and prepares the pupils for per cents over 100 as they occur in practical affairs. Such problems may be solved in either of two ways as may be shown by solving the latter part of the seventh problem. The pupil may find what per cent Harry's weight is of John's by dividing 82 by 65 and carrying the quotient to two decimal places as shown at the left below. Since the quotient is 1.26, Harry's weight is 126% of John's. He may also solve the problem by finding the excess of Harry's weight over John's, finding what per cent this excess is of John's weight, and adding the per cent of excess to 100 as shown at the right below. The excess is 82 pounds less 65 pounds, or 17 pounds. It is seen that 17 pounds is .26 or 26% of John's weight. So Harry weighs 26% more than John, or, his weight is 126% of John's weight. The former solution is ordinarily the better but both should be used for the interpretation represented

by the one tends to supplement and clarify the interpretation represented by the other.

$$\begin{array}{r} 1.26 = 126\% \\ 65\overline{)\,82.00} \\ \underline{65} \\ 170 \\ \underline{130} \\ 400 \\ \underline{390} \\ 10 \end{array} \qquad\qquad \begin{array}{r} .26 = 26\% \\ 65\overline{)\,17.00} \\ \underline{130} \\ 400 \\ \underline{390} \\ 10 \end{array}$$

Teaching the third type. It has been indicated that percentage problems of the third type are of much less frequent occurrence than are those of the first type or the second type. Authors of textbooks, in their efforts to provide practice material of this type, sometimes include problems which are very unreal. The following problems are typical of those which textbooks sometimes contain.

1. Ten per cent of the land which Mr. Smith owns is 16 acres. How many acres does he own?
2. There are 160 boys in our school. This is 40% of all the pupils. How many pupils are there in the school?
3. At the end of the year, Ralph's weight was 88 pounds. This was 110% of his weight at the beginning of the year. How much did he weigh at the beginning of the year?

Each of these problems is unreal, for in each case the answer had to be known before the problem could be written. In the world of practical affairs we do not solve problems to find answers which we already know.

Because some schools seem to have been unable to

find a real and vital need for the third case of percentage, some teachers and writers have suggested that this type of percentage problems be eliminated entirely. It should be eliminated if we can find no needs more vital than those suggested by the three problems which have been given to illustrate the kind which textbooks sometimes contain. However, better problems can be found.

Problems whose solution involves the use of Case III of percentage occur frequently in the affairs of merchants who figure profits on the selling price. Thus, a merchant pays $3.00 each for men's hats. At what price must he sell them so as to make a profit of 40% of the selling price? Since the profit is to be 40% of the selling price, the cost, $3.00, is 60% of the selling price (making no allowance for overhead). In other words, the problem, $3.00 is 60% of what amount?, is a Case III problem.

Goods which are marked so as to permit a designated discount and still leave a given per cent of profit also produce Case III problems. For example, it costs a publishing company $1.00 to publish a book upon which a profit of 50% of the cost is desired. What price should be placed upon the book if a discount of 25% of the marked price is to be allowed and leave a profit of 50% of the cost? Since the profit is to be 50% of the cost, the book must sell for $1.50. If a discount of 25% of the marked price is allowed, $1.50 will be 75% of the marked price. This is clearly an example of the third type of percentage problem. Such a problem is rather involved, however, and probably is too difficult for sixth grade pupils. It may be included in the course for the junior high school.

Problems of the third type are sometimes found in situations which are real from one point of view but unreal from another. For example, a grocer sells potatoes at \$2.00 a bushel and says that he makes a profit of 25% of the cost. What does he pay per bushel for potatoes? In this case, \$2.00 is 125% of the cost of a bushel of potatoes and again we have a problem of the third type. This problem is unreal to the grocer, for he must know the cost before he can compute what the selling price shall be, but it is real to the grower who has potatoes to sell and who knows only the facts which the problem supplies.

In teaching pupils to solve Case III problems, either of two procedures may be employed. The first is the method of unitary analysis. Since 60% of the selling price in the problem about the hats is \$3.00, we may write the solution as follows:

60% of the selling price = \$3.00
1% of the selling price = $\frac{1}{60}$ of \$3.00 = \$0.05
100% of the selling price = 100 × \$0.05 = \$5.00

Or, since 60% is equal to the common fraction, $\frac{3}{5}$, we may write:

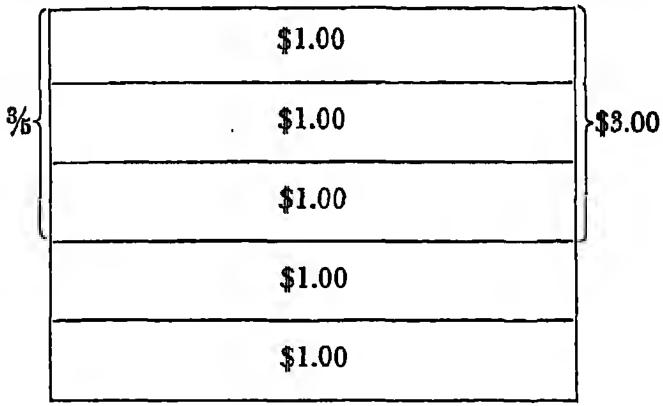
$\frac{3}{5}$ of the selling price = \$3.00
$\frac{1}{5}$ of the selling price = $\frac{1}{3}$ of \$3.00 = \$1.00
$\frac{5}{5}$ of the selling price = 5 × \$1.00 = \$5.00

When pupils undertake a solution by this method, particularly when the common fraction equivalent of the given per cent is used, they often write, for the second step above,

$\frac{1}{5}$ of the selling price = $\frac{1}{5}$ of \$3.00 = \$0.60

This error is due to the failure of the pupils to understand fully the conditions of the problem and to their failure to see the relation of $\frac{1}{5}$ to $\frac{3}{5}$, to see that $\frac{1}{5}$ is $\frac{1}{3}$ of $\frac{3}{5}$. A diagram is often helpful. By its aid, pupils may see that three of the fifths represent $3.00, that one of them must represent $1.00, and that all five of them would represent $5.00.

FIGURE 10. ILLUSTRATION OF THE RELATION OF 60 PER CENT OF THE SELLING PRICE TO THE ENTIRE SELLING PRICE.



If the second method is used, the pupil is taught simply that he can find 100% of anything by dividing the amount given by the per cent given, changing the per cent to a decimal fraction. The solution for this problem would appear as shown. Of course, we can also change the per cent to a common fraction and divide, making the work appear as follows.

$$\begin{array}{r} \$5 \\ .60\overline{)\ \$3.00} \\ \underline{3\ 00} \end{array}$$

$$\$3.00 \div \tfrac{3}{5} = \overset{1.00}{\cancel{\$3.00}} \times \frac{5}{\underset{1}{\cancel{3}}} = \$5.00.$$

This method is hard for pupils to understand. The pupil can be reminded that he always has found the cost of one by dividing the cost of a group by the number in the group. Thus, if 6 pencils cost 30 cents, 1 pencil costs 30 cents ÷ 6, or 5 cents. So, if 3/5 of the selling price of the hat is $3.00, the selling price must be $3.00 ÷ 3/5, or $5.00. Then, to find the selling price of one, divide the amount of the price given by the part given.

The best procedure seems to be to employ first the method of unitary analysis and then to show the pupils that they can get the same result by division. If pupils understand the method of unitary analysis, they will understand well enough for present purposes the second procedure. However, the difficulty of Case III suggests that it may well be delayed for some time after the other two cases have been studied.

Pupils' errors in percentage. An analysis of the errors in percentage made by 405 seventh grade pupils has been reported by Brueckner.[2] In all, 11,735 errors and omissions were listed. Of these, 7,493 were percentage errors and the remaining 4,242 were other errors, such as errors in computation. Omitted items were included in the 4,242 other errors.

Part I of Brueckner's test requires the pupil to change decimal fractions to per cents. On this part, there were 1,770 percentage errors. Most of these errors were instances of simply adding the per cent sign to the number given and dropping the decimal point, if any, re-

[2] Brueckner, Leo J. *Diagnostic and Remedial Teaching in Arithmetic.* Philadelphia: The John C. Winston Company, 1930, pp. 242-257.

gardless of the number of decimal places which the number had. It was evident here that there was a marked lack of understanding of the relationship of percentage to decimal fractions.

Part 2 of this test consists of exercises in changing per cents to decimal fractions. Here there were 1,034 errors. Again, there was clear evidence to the effect that these pupils did not understand the relationship of percentage to decimal fractions. They dropped the per cent sign without changing the position of the decimal point, they moved the point an incorrect number of places, they inserted zeros, and they made other like errors.

The third and fourth parts of the test require that common fractions be changed to per cents. There were on these two parts a total of 1,239 percentage errors. Conspicuous here was the fact that these pupils did not know well the common fraction equivalents of per cents and again that they did not know the relationship of decimal fractions to per cents.

The fifth part of the test makes use of Case I. Here, there were only 340 percentage errors. Again it was evident that the fundamental meaning of percentage was not understood.

Part 6 uses Case II of percentage. This part accounted for 2,049 percentage errors. Indeed, not one of the nine items in this part of the test was correctly solved by as many as 40 per cent of the pupils. There were 1,341 instances of dividing the base by the percentage instead of dividing the percentage by the base to find what per cent one number is of another. At times, the base was *multiplied* by the percentage. There were a number of

instances of dividing correctly but failing correctly to express the quotient as a per cent.

The last part of the test is made up of items in Case III of percentage. There were 1,061 percentage errors on this part. The easiest item in this part was solved correctly by only 21.7 per cent of the pupils. Only 3.5 per cent of these pupils solved correctly the item, 255 = 125% of ——. There were many kinds of errors but most of them indicated that the pupils lacked much of understanding what is to be done in Case III. For that matter, they lacked much of understanding percentage at all, as has already been indicated.

Whether these results are typical of the results which would be secured by other analyses can not be stated. But the author's observations have led him to believe that much of the difficulty which pupils experience with this subject is due to the prevalence of the drill theory of instruction and consequent rote learning. A thoroughgoing reform in percentage teaching is needed. The best single suggestion as to the direction which this reform should take is that instruction should be based upon the meaning theory. In the preceding pages an effort has been made to incorporate this theory in the suggestions which have been offered.

Just recently, a seventh-grade teacher in describing to the author her efforts to teach Case II of percentage remarked that her pupils persisted in dividing the larger number by the smaller in undertaking to find what per cent one number was of another regardless of which number was the base. She found that one reason for this was the fact that in preceding grades the teachers of these pupils had insisted vigorously that in dividing one

always should divide the larger of two numbers by the smaller. Apparently, there had been no explanation of why this should be done. The pupils had been given a mandate and they had obeyed blindly and quite without understanding.

The applications of percentage. As pupils study the three cases of percentage, they should have many meaningful applications of these cases to that portion of the world of business affairs with which they are acquainted. These applications make the subject of percentage much more interesting to pupils and they add greatly to their understanding of it. Applications should be woven into the treatment of each case as it is presented rather than postponed until after all of the basic processes have been studied in abstract form.

In this chapter, we consider briefly four common applications of percentage. They are: (1) discount; (2) commission; (3) profit and loss; and (4) interest.

As these applications of percentage are studied, pupils will not only find opportunities to use what they are learning about percentage but also they will acquire considerable information of social value. The social aspects of the subject are emphasized in the following pages.

Discount. The subject of discount is of interest to practically all adults and to many children. Newspapers frequently contain advertisements which carry such announcements as "1/3 off," "25% off," etc. Such advertisements are an excellent source of material for teaching the meaning and more common uses of discount and for developing in the pupils a better understanding of percentage.

Pupils will not only have practice in the arithmetic of discount but also they will learn the reasons why articles are sold at reduced prices. They note that some articles, especially articles in women's apparel go out of style. Others are sold at a reduction because of the expense of storage and insurance and the danger that they may be damaged by moths or rust. Other articles become shopworn and still others must be sold to make room for new stock. Also, merchants sell at a discount as a means of stimulating trade in a dull season, or as a part of an advertising campaign, or to raise money to meet bills which are due. Of course, merchants also allow small discounts for cash rather than incur the expense and the risk of opening and carrying accounts.

Most discount problems are problems in the first case of percentage or its equivalent in common fractions. The following are typical.

1. A shoe dealer offers a 20% discount on shoes bought during a sale. What will be the cost of a pair of shoes if the price was $4.95?

2. Girls' hats are advertised at ⅓ off. What will a $4.50 hat cost?

3. "All men's shirts reduced 25%" reads an advertisement. What will be the price of a shirt which was marked $3.00?

Sometimes, second case problems in percentage are provided by discounts. One may wish to know what per cent he saves in a marked-down sale where the rate of discount is not given. This is illustrated by the following problems.

1. A boy's suit is marked down from $19.95 to $15.50. What per cent of the original price is saved?

2. Berries were selling at 28 cents a quart but on Saturday evening they were marked "19 cents." What per cent of the original price is saved?

3. If five dollar shoes are marked down to $3.98, what per cent of the original price is saved?

Commission. Commission is of less general interest than is discount. That is, fewer persons have experiences with commission than with discount. But at some time in their lives many adults and also many children have experience with commission in some form. Boys and girls are frequently unwelcome callers as they undertake to sell articles or to take subscriptions for a commission in a house-to-house canvass.

The older textbooks contained commission problems which had been drawn entirely from the experiences of a few adults. They told about commission merchants who bought and sold grain, wool, lumber, etc., in large quantities. The deals frequently involved large sums of money—large to children at any rate. The present tendency to draw problems from activities in which the children themselves engage or in which their parents have had a part is much better. The following problems will suggest the kinds which may be used.

1. Alice sold 28 tickets for the high school play at 25 cents each. The principal said, "We will pay you a commission of 10%." How much did Alice earn?

2. Jack sells the Saturday Evening Post. Last week he sold 33 copies at 5 cents each. He was allowed to keep 2 cents from each sale for himself. How much did he make?

What per cent of the money did he keep for himself?

3. Mary Elizabeth sells Christmas cards and seals at 10 cents a package. She keeps 40% of the money for herself. How much does she keep if she sells 50 packages?

4. John's father sold his house for $7,200. He paid the real estate man 2% for making the sale. How much did the real estate man receive?

5. Ruth's older brother sells brushes to make money to go to college. The company allows him 30% of all his sales. In August, his sales amounted to $426.50. How much did he make?

6. George is allowed a 10% commission for selling music course tickets at $3.00 each. How much does he make if he sells 25 tickets?

Most of the problems in commission will be problems of the first type. A few, as the second part of No. 2 above, will be Case II problems. Most of them can be drawn from activities in which children participate but a few, as the fourth and the fifth in the above list, may refer to the affairs of their elders.

Profit and loss. At times, most persons are interested in the subject of profits and losses. This topic largely represents an aspect of business as it is found in the adult world rather than in the world of children but it has some interest to children not only because they hear of it from their elders but also because it is found now and then in their own affairs. In the business world, profit and loss is an important application of percentage. The arithmetic course of study should make some provision for problems in profit and loss to illustrate another use of percentage, to give further practice in the cases of percentage, and to supply information about business practices.

In business practice, the difference between the cost of an article and its selling price is not called *profit* but *margin*. The profit is what is left of the margin after the expenses of selling are subtracted. Thus, if a merchant pays \$20 for a suit of clothes and sells it for \$32, the margin is \$12. Since \$12 is 37½ % of \$32, 37½% is the *per cent of margin*. The per cent of margin is based on the selling price. The difference between the selling price and the cost is also called the *advance*. Here, the advance in cost is \$12. Since \$12 is 60% of \$20, 60% is the *per cent of advance* over the cost. The per cent of advance is based on the cost.

Now, if the expenses of selling this suit of clothes (rent, advertising, insurance, taxes, wages, heat, light, etc.) amount to \$6.25, there is left a profit of \$5.75. Since \$5.75 is approximately 18% of \$32, 18% is the *per cent of profit*. The per cent of profit is based on the selling price. If there is a loss, the per cent of loss is also based on the selling price. Expenses of selling are often referred to as *overhead*.

The distinction between *margin* or *advance* and *profit* is one which pupils should know and understand eventually but in their first work in the subject it is better to simplify the problems by leaving out of consideration for a time the matter of expenses of selling. If a child buys an article and later sells it to a playmate for more than he paid for it, he may very properly call the difference between the cost and the selling price *profit* for, presumably, he has no selling expenses. Then, he may find what per cent this profit is of both the selling price and the cost. Of course, he should understand that if a merchant says that his profit on a sale is 20 per cent, he

usually means that his profit is 20% of the selling price and that he has allowed for selling expenses.

The problems which are solved may represent a wide variety of child and adult experiences. The following are typical.

1. William bought a second-hand bicycle for $4.00. He bought two new tires for $3.00 each and paid 25 cents for paint. He then sold the bicycle for $12.00. What per cent of the $12.00 was profit? What per cent of the total cost did he gain?

2. Alice paid $2.00 for a puppy. Three months later a man offered $10.00 for him and Alice sold him. "Just think," said Alice, "I made $8.00." But her father said, "You have forgotten what it cost to feed him, haven't you?" They estimated that it had cost $3.00 a month to feed him, not counting scraps that would have gone into the garbage can. Did Alice really gain or lose? How much? What per cent of the total cost? What per cent of the selling price?

3. The next day, Alice remembered that she had forgotten to count the $1.00 which she had paid for a license for her puppy. So she figured again what she had lost and what per cent of the total cost she had lost. What were her answers?

4. Last summer Jack agreed to mow the lawns of several of the neighbors once a week for 50 cents a week each. One week he was so busy that he had to hire Ted to mow six of the lawns. Jack said, "I will keep 20% of what Ted earns for myself." How much did Jack make from Ted's work? How much did Ted receive?

5. Eleanor buys Christmas seals at $4.00 a hundred packages and sells them at 10 cents a package. What per cent of the cost does she make?

6. One afternoon last summer, Harry and Fred decided

to have a lemonade stand. They bought one dozen lemons for 50¢ and 6 pounds of sugar at 7 cents a pound and paid 25 cents for ice. They sold 9 glasses of lemonade at 5 cents a glass and then tried to sell the rest at the rate of two glasses for 5 cents but sold only 10 more glasses. Did they gain or lose? What per cent of the cost did they gain or lose? What per cent of the selling price?

7. A grocer paid $1.25 a bushel for potatoes. At what price per bushel should he sell them to make an advance of 25% on the cost?

8. Mr. Jackson bought a house for $5350. He paid $62.40 taxes, $15.75 for insurance and $122.50 for repairs. Then he sold the house for $5750. What per cent of the total cost did he gain? What per cent of the selling price did he gain? Why is the second answer smaller than the first?

9. A dealer paid $6.50 per pair for men's shoes. For how much must he sell them to make a profit of 35% of the selling price?

10. In the summer Dick works as a caddy at the city golf course. The caddies get 10 cents each for the golf balls they find. Dick bought 35 balls from the other caddies at 10 cents each, repainted them and sold them for 25 cents each. If the paint cost him 50 cents, what per cent of the total cost did he make?

Most of the profit and loss problems which the average man has to solve involve finding the per cent gained or lost on some transaction; in other words, they are Case II problems. However, it sometimes happens that Case I problems and Case III problems must be solved also. In the above list, problems 4 and 7 represent the first type and problem 9 represents the third type.

Interest. There are few adults who do not pay or col-

lect interest. Interest paid may be concealed in a so-called "carrying charge" when merchandise is purchased on a deferred-payment plan but it is there nevertheless and the rate is usually rather high. Interest paid on loans from banks, building and loan companies, and individuals is of importance to adults but of little significance to children; but the interest paid on postal savings deposits, "baby bonds," and accounts in savings banks is of concern to children as well as to adults, for many boys and girls have their own interest-bearing accounts. Warren and Burton found that 48.7 per cent of the 1050 pupils whom they studied at Long Beach, California, had had experience depositing money in the bank and that 18.8 per cent had made deposits in building and loan banks. The per cent which had savings accounts was found to be 74.3.[8] These per cents may be higher than the per cents found in other communities but they do indicate that there is in the experiences of many children a basis for lessons in the percentage of interest.

The average person who borrows or lends money is concerned with finding the amount to be paid as interest but not with finding the rate or the principal. In other words, interest problems are largely problems in the first case of percentage. Textbooks have contained too many problems in which the interest and the rate were given and which required that the principal be found. Such problems are nearly always unreal.

The older textbooks frequently contain problems which require pupils to find interest for long and un-

[8] Warren, Dorothy E. and Burton, W. H. "Knowledge of Simple Business Practices Possessed by Intermediate-Grade Pupils." *Elementary School Journal,* XXXV: 511-516, March, 1935.

usual periods of time. In actual business affairs, however, interest is usually paid at the expiration of such periods of time as 30 days, 60 days, 90 days, 3 months, 6 months, or 1 year. Very rarely, indeed, is interest computed for such a period as 1 year, 5 months, and 17 days. Outside of an occasional instance when an account is collected through the courts and interest is allowed from the date the account was due to the date of settlement, interest is nearly always paid at the end of a previously designated period of time, such as 60 days, 6 months, etc.

In their first work in interest, pupils should learn about interest as a social institution and should see an application of Case I in percentage. Short methods should come later. Pupils should easily see that the interest on $250 for one year at 6 per cent is .06 of $250, or $15.00. For six months, the interest is one-half of this, or $7.50. For four months, it is one-third of this, or $5.00. For three months, it is one-fourth of $15.00, or $3.75. If the period of time is 3 months, the work may appear as shown. The pupil should easily see that he should divide $15.00, the interest for one year, by 4, since 3 months is one-fourth of a year.

$$
\begin{array}{rl}
\$250 & \\
.06 & \\
\hline
4\,)\,\underline{\$15.00,} & \text{interest for 1 year} \\
\$3.75, & \text{interest for 3 months}
\end{array}
$$

If the period of time is given in days and the number of days is not an easily handled fraction of a year, it seems best to use the form of solution which involves the multiplication of common fractions. If the interest

on $400 for 48 days at 6 per cent is required, the solution may appear as shown. However, such work should be postponed until there has been considerable practice on easier examples.

$$\frac{\overset{4}{\cancel{48}}}{\underset{\underset{5}{\cancel{60}}}{\cancel{360}}} \times \frac{\overset{1}{\cancel{6}}}{\underset{1}{\cancel{100}}} \times \$\overset{4}{\cancel{400}} = \frac{\$16}{5} = \$3\tfrac{1}{5} = \$3.20$$

The following problems are typical of those which may be used.

1. Richard saves his money and puts it in the Citizen's Savings Bank. The bank pays 3% interest. How much interest should Richard get if he leaves $45 in the bank for a year?

2. Many savings banks pay interest every six months for money which is left in the bank the entire six months. Ralph had $30 in the bank on January 1. How much interest should he receive on July 1 at 3%?

3. The building and loan company in our city pays 4 per cent interest on money left with them from January 1 to July 1, or from July 1 to January 1. Ralph's father deposited $1500 with the building and loan company on January 1. How much interest should he receive on July 1? If he deposits the interest and $300 more on July 1, how much interest should he receive on January 1?

4. When Mr. Botts bought a house, he borrowed $4000 at 6 per cent and agreed to pay the interest and $200 of the principal every six months. How much did he owe at the end of the first six months? At the end of the second six months?

While pupils are learning to solve problems in in-

terest, of course they should learn the new terms used, such as *interest, principal,* etc. These terms are best learned through a description of a real or a hypothetical case. The explanation of the meaning of *interest* and *principal* may be made in the following manner.

"When Mr. Brown started in business for himself he borrowed $2,500 from his friend, Mr. Briggs, for one year. Mr. Brown agreed to pay Mr. Briggs the $2,500 at the end of the year and he also agreed to pay him 6 per cent of the $2,500 for the use of it for a year. The extra money which Mr. Brown paid Mr. Briggs for the use of the $2,500 is called *interest*. The $2,500 is called the *principal*. To find how much interest Mr. Brown owed Mr. Briggs at the end of the year we find 6 per cent of $2,500, or, multiply $2,500 by .06, since per cent means hundredths. Then at the end of the year, Mr. Brown owed Mr. Briggs $2,500, the principal, plus $150, the interest, or $2,650 in all."

$$\begin{array}{rl} \$2500, & \textit{principal} \\ \underline{.06} & \\ \$150.00, & \textit{interest} \end{array}$$

The early use of percentage. Ordinarily, the study of percentage follows the study of decimal fractions. It is right that it should but it is not necessary that the subject of decimals be completed before the study of percentage is begun. The first case of percentage is easier than division with decimals and it is of far greater practical value. Then, the idea of percentage may well be introduced before division with decimals is completed.

If illustrations of the uses of percentage are drawn from affairs which children experience and of which they know, percentage from the start may be both meaningful and interesting. Sometimes, a game or a group

enterprise affords a means of adding to interest in the subject and the pupils' understanding of it. In one class, for example, the pupils collected from their homes toys and other playthings for a second-hand sale. When the articles had been brought to school, they were listed on the blackboard with their prices when new and the second-hand prices which were to be charged. The following list is typical of the items which were listed. After the costs and second-hand prices had been recorded, the pupils figured for each the per cent which the second-hand price was of the cost.

	Cost	Second-hand Price	Per Cent
1. Boy's bicycle	$32.00	$8.00	25
2. Girl's bicycle	36.00	5.00	14
3. Ball glove	1.50	.50	33
4. Ball bat	.50	.30	60
5. Doll	2.00	.25	12
6. Game	.75	.20	27
7. Scooter	6.00	2.50	42
8. Roller skates	2.50	.75	30
9. Ice skates	1.75	.75	43
10. Doll's dishes	.98	.25	26
11. Football	1.95	.60	31
12. Ukelele	3.25	1.25	38

If the processes of percentage are to be understood and if pupils are to retain what they learn from the study of it, it must be taught as a part of a series of interesting and meaningful experiences. The mere memorizing of terms and definitions and the solving of problems by rule and in imitation of set models means that the pupils will learn but little and will retain

permanently almost nothing. As a consequence, they will find themselves unable to solve the practical problems of the every-day world in which percentage is involved. Pupils not only learn more easily that which they understand and enjoy but also they retain it and use it with a greater degree of success.

QUESTIONS AND REVIEW EXERCISES

1. Do you believe that percentage should be difficult for pupils in grades six and seven? How do you account for the many evidences of misunderstanding which are found when pupils are tested on percentage?

2. Examine the sport pages of a newspaper for standings of baseball teams in the major leagues. Are the so-called per cents given to two or three decimal places? If per cent means hundredths, can it also mean thousandths?

3. What benefit is there in studying the etymology of new words as these words are learned in school? Is it worth while to present the Latin origin of words to pupils who have not studied Latin?

4. What are the four fundamental processes in percentage? Which of these processes do you believe to be the most important? The least important?

5. What use would you make of the rule: To change a decimal fraction to per cent, move the decimal point two places to the right and affix the per cent sign?

6. What would you say is a pupil's difficulty if he says, $\frac{1}{4} = .25 = .25\%$? What would you do to correct this error?

7. Are there any fractions in the list of aliquot parts whose decimal equivalents you would not have pupils memorize? Are there any which you would add to this list?

8. Can you count rapidly and smoothly by per cents to 100 per cent using sixths and eighths? Is it reasonable to expect pupils to learn to do this?

9. In the practical uses of arithmetic, what are the circumstances under which pupils should change per cents to common fractions?

10. State some situations which are likely to require the use of per cents greater than 100. How do you account for the fact that per cents greater than 100 are often difficult for pupils to understand?

11. State the three cases of percentage. Which of these is the most frequently used in business affairs? Which is least frequently used?

12. In a percentage example, which number represents the percentage? Illustrate.

13. In undertaking to solve the example, What per cent of 16 is 4?, several pupils divided 16 by 4 and announced 4% as the answer. What is the probable nature of the difficulty which these pupils had? What would you do about it?

14. An account of a safety campaign for school children states that the result was a reduction in accidents of several hundred per cent. What does this mean?

15. What is the objection to the use of problems of the "answer-known" type to teach the third case of percentage?

16. Is Case III of percentage of sufficient importance to justify its inclusion in the course of study? Does the teaching of this case serve any useful purpose in addition to that of enabling pupils to cope with business situations in which it is found?

17. What is the method of unitary analysis? What difficulties is the teacher likely to experience in teaching this method? How can these difficulties be overcome?

18. What were the outstanding types of errors reported by Brueckner? What should the teacher do to overcome these errors? In general, would you say that this error study indicates that percentage had not been well taught?

19. What cases of percentage are used in solving problems

in discount? Which case is used most frequently? Is the subject of discount of interest to children?

20. Are children interested in commission? What case in percentage is usually employed in solving commission problems?

21. What is the profit in a business transaction? How does it differ from the margin?

22. A dealer buys shoes for $3.40 per pair and sells them for $5.00. He estimates the expenses of selling a pair of shoes to be 72 cents. What is his per cent of margin? His per cent of advance? His per cent of profit?

23. What method of calculating interest would you teach first? When should such short methods as the "60-day, 6 per cent" method be taught? When should pupils be taught to use interest tables?

24. Do twelve-year-old children have enough experience with interest to justify including this subject in the course of study?

25. How early should the elements of percentage be taught? How can the teacher avoid rote learning in this subject? Do you believe that percentage can be learned satisfactorily if a method based upon the theory of incidental learning is used?

CHAPTER TEST

For each of these statements, select the best answer. A scoring key will be found on page 534.

1. Per cent means (1) per hundred (2) hundredths (3) by the hundred.

2. When pupils begin the study of percentage, they should be impressed with the fact that (1) it is a new idea (2) it is a familiar idea expressed in familiar language (3) it is a familiar idea expressed in new language.

3. The number of fundamental processes in percentage is (1) 3 (2) 4 (3) 5.

4. The per cent sign, %, is equivalent to (1) 1 (2) 2 (3) 3 decimal places.

5. Pupils should learn to change common fractions to decimal fractions (1) before they study percentage (2) while they are studying percentage (3) after they study percentage.

6. The fraction, $\frac{1}{12}$, is equal to (1) 8 per cent (2) 83 per cent (3) $8\frac{1}{3}$ per cent.

7. The list of fractions whose per cent equivalents are to be memorized does not include (1) $\frac{5}{6}$ (2) $\frac{3}{8}$ (3) $\frac{1}{12}$.

8. If a pupil's score on a test increases from 10 to 30, his per cent of increase is (1) 200 (2) 300 (3) $66\frac{2}{3}$.

9. In Case I of percentage, the quantity to be found is (1) the base (2) the rate (3) the percentage.

10. In Case II of percentage, the quantity to be found is (1) the base (2) the rate (3) the percentage.

11. In Case III of percentage, the quantity to be found is (1) the base (2) the rate (3) the percentage.

12. The most frequently used case of percentage is (1) Case I (2) Case II (3) Case III.

13. The least frequently used case of percentage is (1) Case I (2) Case II (3) Case III.

14. If a common fraction is changed to per cent, the quotient should be expressed to (1) the next lower hundredth (2) the nearest hundredth (3) the next higher hundredth.

15. If it is known that *a* per cent of a number is *b*, the number, *n*, is equal to (1) b/a (2) a/b (3) ab.

16. According to the findings of the error study reported in this chapter, the most serious errors were (1) errors in computation (2) errors in spelling (3) errors due to a failure to understand the meaning of percentage.

17. Instruction in percentage should be based upon (1)

the meaning theory (2) the drill theory (3) the theory of incidental learning.

18. The percentage of discount is largely percentage of (1) Case I (2) Case II (3) Case III.

19. The second most frequently used case in the percentage of discount is (1) Case I (2) Case II (3) Case III.

20. The profit is (1) selling price minus cost (2) cost minus selling price (3) selling price minus expenses minus cost.

21. The per cent of profit is based on the (1) cost (2) selling price (3) margin.

22. Short methods of calculating interest should be taught (1) before the longer methods (2) along with the longer methods (3) after the longer methods.

23. In an interest problem the percentage is (1) the principal (2) the interest (3) the rate.

24. The first work in percentage should come (1) before decimals are finished (2) after decimals are finished (3) before decimals are studied.

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2. Edwards, Arthur. "A Study of Errors in Percentage." National Society for the Study of Education, *Twenty-Ninth Yearbook*. Bloomington, Illinois: Public School Publishing Company, 1930, pp. 621-640. Presents the results secured from an analysis of the papers of 215 seventh-grade pupils.

3. Glazier, Harriet E. *Arithmetic for Teachers*. New York: McGraw-Hill Book Company, 1932, 291 pp. The

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7. Warren, Dorothy E. and Burton, W. H. "Knowledge of Simple Business Practices Possessed by Intermediate-Grade Pupils." *Elementary School Journal*, XXXV: 511-516, March, 1935. Presents the results of the use of a check list of items with 1050 fifth- and sixth-grade pupils.

8. Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic*. Boston: D. C. Heath and Company, 1937, 591 pp. An introductory discussion of percentage will be found on pages 96-101. The subject appears again as a part of a chapter entitled "The Study of Parts" on pages 418-422.

CHAPTER 9

DENOMINATE NUMBERS

Development of weights and measures. The two systems of weights and measures in most common use in the civilized world of today are the English system and the Metric system. The English system was brought to this country early in our history and, with modifications, is with us at the present time. The metric system, adopted first by the French government in 1799, has since been officially adopted by all of the important nations except Great Britain and the United States, although it is not uniformly used in the countries which have adopted it officially.

The English system is a conglomeration of irregularities. To transform one unit of measure to another, such numbers as 2, 3, 4, 5½, 8, 9, 12, 16, 16½, 24, 27, 30¼, 36, 60, 144, 160, 231, 320, 360, 640, 1728, 2150.42, 7000, and others, are used. The metric system, on the other hand, uses only the number 10 and its powers, especially the second and third powers, 100 and 1000. It has a tremendous advantage in its simplicity and the ease with which it may be used but custom and conservatism prevent its general adoption. Those in this country who admire our metric system for money and appreciate its convenience and who smile at the English system with its farthings, pence, shillings, and pounds, refuse to take seriously the suggestion that we might enjoy similar advantages in measures of weight, distance, area, and capacity if we would only adopt the metric system for such measures.

Although our system of weights and measures is essentially the English system, there are differences between some of the measures employed in this country and those used in Great Britain. Our gallon, for instance, is the old English Winchester gallon and contains 231 cubic inches but the gallon in use in Great Britain is the imperial gallon which contains 277.274 cubic inches. Likewise, the American official bushel, the Winchester bushel, contains 2150.42 cubic inches while the English imperial bushel contains 2218.192 cubic inches. The imperial bushel is the volume of 80 pounds of water at 62 degrees Fahrenheit and is just eight times the imperial gallon. It will be noted that no such relation exists between the Winchester gallon and the Winchester bushel. The Winchester bushel has been a standard measure since Anglo-Saxon times. The standard bushel measure was preserved in the town hall at Winchester.

Early efforts at measuring distance were expressed largely in terms of parts of the human body and in terms of human performance. Thus, the length of the foot, the length from the elbow to the tips of the fingers (the cubit), the length of the arm (the yard), the total arm stretch (the fathom), the width of the hand, the span of the hand, and the length of the thumb joint, became units for measuring short distances. For greater distances, there was the pace, the bowshot, the day's journey, and others. Some of these measures are still in fairly common use and we also have the five minutes walk, the half-hour's ride, the mountaineer's "two

whoops and a holler," and others.[1] And there is the handful of grain, the armful of packages, the pinch of salt, the dash of pepper, and other crudely descriptive measures of volume.

Such measures served a useful purpose but, obviously, their defects lay in the fact that they varied from person to person and from place to place. If a short-armed merchant is selling cloth to a long-armed customer it makes a real difference as to whether the merchant's arm or the customer's arm is employed as the measure of length. For a long time society struggled with the task of standardizing the basic units of measure. Imperial edicts, legislation, conferences, and international agreements mark the progress which man has made in developing and agreeing upon standard measures. Thus, the inch at one time was the total length of three barleycorns taken from the middle of the ear and well dried; the rod was suggested to be the total length of the left feet of 16 men chosen at random as they left church on Sunday morning and the foot was one-sixteenth of this rod, or the average of several feet; the lawful yard became the distance from the end of the nose of King Henry I to the end of his thumb; Edward III is said to have had the length of his arm recorded by a metal bar which was divided into three equal parts, each of which was divided into twelve equal parts; the grain became the weight of a grain of wheat; the penny or sterling became the weight of thirty-two grains and twenty

[1] Cf. Sanford, Vera. *A Short History of Mathematics*. Boston: Houghton Mifflin Company, 1930, p. 353.

pence became an ounce and twelve ounces a pound; eight pounds became a gallon and eight gallons a bushel; etc.

The origin of many of our measures is indicated by their names. Some of these are summarized briefly in Table 7.

TABLE 7. SOME COMMON NAMES OF MEASURE AND THEIR PROBABLE ORIGINS

Measure	Probable Origin
Inch	*Uncia,* a twelfth part
Foot	Human foot
Yard	Arm length
Pace	A runner's step (Roman); about five feet
Mile	*Millia passuum,* 1000 paces
Grain	A grain of wheat
Ounce	*Uncia,* a twelfth part (also one-sixteenth)
Gill	*Gilla,* a small drinking glass
Pint	*Pinta,* a mark (on a larger vessel)
Quart	*Quartus,* one-fourth (of a gallon)
Peck	A quantity easily carried (pack)
Bushel	*Bussula,* a small box
Minute	*Minutem,* a small part
Second	*Secundus,* a second small part
Hour	*Hora,* time or period of time
Month	A moon (time for a revolution of the moon)

It may be noted that various number systems have left their traces in the tables of denominate measures. Primitive units were subdivided decimally in Egypt and also in China and Japan, they were divided duodecimally by the Romans and sexagesimally among the Babylonians and the Assyrians. Binary subdivisions (½, ¼, ⅛, etc.) were found among the Hindus and later among the Germanic and Teutonic peoples. Today, we very commonly have decimal divisions of an inch in machine shop practices (tenths, hundredths, thousandths, etc.). We have traces of the duodecimal as indicated by the fact that there are 12 inches in a foot, 12 ounces in a troy pound, and 12 things in a dozen. We have the sexagesimal in seconds in a minute and minutes in an hour, or degree. And, of course, binary subdivisions are very commonly applied to inches, yards, miles, gallons, pints, hours, and other measures.

When should denominate numbers be taught? At one time, denominate numbers were taught as a separate subject in arithmetic, that is, separate from other subjects, in a chapter by themselves. During the weeks devoted to this subject, the pupils memorized many tables, they added, subtracted, multiplied, and divided compound denominate expressions, and they engaged in long exercises in reduction. The present tendency is to break up this subject, teaching a little here and a little there as needed and as it fits into other subjects.

Thus, the fact that there are two pints in a quart will be learned early and will be used in making problems for practising the twos in multiplication. Such questions as, How many pints in four quarts? Two quarts? Five quarts? Three quarts? etc., not only help to furnish

variety in providing practice on twos but they also help greatly in fixing the fact that a quart contains two pints. Likewise, we may use the fact that there are three feet in a yard in practising the threes; that there are four quarts in a gallon and four pecks in a bushel in practising fours; that there are five cents in a nickel in practising fives; that there are seven days in a week in practising sevens; etc. Such problems make both multiplication and denominate numbers more interesting and help pupils to remember what they are learning.

Later, when the necessary facts have been learned, the tables may be memorized as wholes as a review and as a means of tying together the separate items.

Denominate numbers in the primary grades. The study of measures very properly begins in the first grade. In an informal manner first-grade pupils learn to tell the day of the week and the day of the month; the value of our common coins and how to distinguish them; the meaning of pint, quart, inch, foot, dozen, etc.; and to make comparisons between articles to tell which is longer or shorter, larger or smaller, heavier or lighter, etc. This type of activity is continued through the second and the third grades and additional items of information and further skills with measures are acquired.

For the primary grades, the following suggestions as to measures have been offered.[2]

First grade:

(*a*) Time: days of the week; some or all of the

[2] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company, 1937, Chapters 10 and 12.

months; day of the month; yesterday, today, tomorrow; important dates, such as Thanksgiving, Christmas, Valentine Day, and birthdays.

(*b*) Length: inch, foot, half-inches; inches in a foot.

(*c*) Liquid: the glass or cup, the pint, and the quart; glassfuls or cupfuls in a pint; pints in a quart.

(*d*) Coins: cent (penny), nickel, dime, quarter; cents in a nickel, cents in a dime, cents in a quarter; the signs, $, ¢.

(*e*) Terms: dozen and half-dozen.

(*f*) Fractions: one-half and one-fourth.

(*g*) Comparisons: larger, smaller, bigger, littler, taller, shorter, longer, shorter, higher, lower, heavier, lighter, nearer, farther, more, less.

Second grade:

(*a*) Time: hour, half-hour, quarter-hour, minute, year; days in a week, months in a year, days in current months.

(*b*) Length: yard, half-inch, quarter-inch.

(*c*) Liquid: gallon, half-gallon; quarts in a gallon; quarts in a half-gallon.

(*d*) Money: dollar, half-dollar; quarters in a dollar; quarters in a half-dollar.

(*e*) Fractions: further practice with one-half and one-fourth as applied to the hour, the inch, the gallon, the dollar, and the pound.

(*f*) Weight: the pound, half-pound, quarter-pound.

Third grade:

(*a*) Length: mile, half-mile, quarter-mile, block or square; feet in a yard, inches in a yard.

(*b*) Weight: ounce.

(*c*) Money: dimes in a dollar, cents in a dollar.
(*d*) Fractions: Use in comparing units of measure, as a pint is one-half of a quart, a quart is one-fourth of a gallon, a foot is one-third of a yard, a cent is one-fifth of a nickel, a nickel is one-fifth of a quarter, a day is one-seventh of a week, etc.; unit fractions for those division facts which are taught; three-fourths and two-thirds.
(*e*) Abbreviations of measures taught.

Denominate numbers in the intermediate grades. Naturally, the measures needed will depend upon the particular environment in which the pupils live. These will vary from one community to another and will be slightly different for those living in urban as contrasted with rural environments. Any selection which may be recommended will be to some extent an arbitrary one. Textbooks frequently contain more than pupils should be expected to learn whereas some important items may have been omitted from the books entirely. Textbooks never contain more than a fraction of all the measures which could be included. A minimum list probably will include the following relations for the intermediate grades.

(*a*) Length: Inches in a foot, inches in a yard, feet in a yard, feet in a mile. For rural communities add: feet in a rod and rods in a mile.
(*b*) Area: Square inches in a square foot, square feet in a square yard. For rural communities add: square rods in an acre, acres in a square mile or section, acres in a quarter-section.

(*c*) Volume: Cubic inches in a cubic foot, cubic feet in a cubic yard.

(*d*) Capacity: Pints in a quart, quarts in a gallon, cubic inches in a gallon, and, in some communities, gills in a pint (liquid measure); pints in a quart, quarts in a peck, quarts in a bushel, pecks in a bushel (dry measure).

(*e*) Weight: Ounces in a pound, pounds in a ton (avoirdupois).

(*f*) Time: Seconds in a minute, minutes in an hour, hours in a day, days in a week, days in the months, days in a year (including leap year), weeks in a month, weeks in a year, months in a year.

(*g*) Money: The United States coins in common use—cent, nickel, dime, quarter, half-dollar, dollar; the more common paper denominations—one, two, five, ten, and twenty dollar bills.

(*h*) Temperature: The Fahrenheit scale; freezing, boiling, and other temperatures.

(*i*) Abbreviations of measures taught.

Other interesting facts will be included from time to time. For example, there are usually 24 quarts in a crate of berries; a dry quart is considerably larger than a liquid quart so we can not say that there are four quarts of berries in a gallon; the gallon used in Canada is almost exactly 20 per cent larger than our gallon, hence 10 of their gallons equal 12 of ours; dollar bills are much used in the eastern part of the United States but in part of the West silver dollars take their place; the old Spanish-American practice of cutting silver dollars to make change finally produced eighths of a

dollar each of which was called a *bit,* hence two bits make a quarter or twenty-five cents; one-half of a bit, one-sixteenth of a dollar, is said to have been called *picayune;*[8] a size 7 shoe is about one-third of an inch longer than a size 6 shoe; when timothy hay has become well settled, it requires about 450 cubic feet to make a ton; etc. The alert and resourceful teacher will find many such interesting and illuminating bits of information.

When possible, such items of information should be taught concretely. The pupil will know better that there are two pints in a quart if he actually fills a pint measure twice and empties it into a quart measure than if he merely memorizes the fact. He should compare the sizes of the liquid and the dry quarts by filling a measure of the former size and emptying it into one of the latter size. He may make a box eleven inches long, seven inches wide, and three inches deep and, using sand, discover that it holds just a gallon; then, he may find the number of cubic inches in the box. Information secured in this manner is far more meaningful than is information secured from the textbook or from the teacher.

Avoid unreal problems and exercises. Courses of study and textbooks formerly contained many unreal problems in reduction and in the operations with denominate numbers. The following were more or less typical.

1. Reduce 1 week, 2 days, 5 hours, 31 minutes, and 23 seconds to seconds.

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[8] This was not the only meaning of *picayune.*

2. Change 72, 461 inches to higher denominations.

3. Reduce 4 gallons, 3 quarts, and 1 pint to gills.

4. Subtract 1 mile, 271 rods, 4 yards, 2 feet, and 8 inches from 3 miles, 65 rods, 4 yards, 2 feet, and 5 inches.

5. Multiply 2 tons, 5 hundredweight, 73 pounds and 11 ounces by 7.

Such exercises reflect the old-time disciplinary point of view in arithmetic teaching. Doing them develops a type of skill which functions to a slight extent, if at all, in solving problems of measuring in a real world. Some reduction and some work in the operations are needed but they can be taught in problems of greater interest and more general appeal than those which have been stated. A teacher who observed the out-of-school interests and activities of her pupils used problems such as these.

1. Geraldine bought a five-cent candy bar. On the wrapper she read, "Net weight, 1¼ oz." At this rate, how much would a pound cost?

2. What is the cost of a pound of "Life-Savers" if a five-cent package contains ¾ of an ounce?

3. If it takes two glasses to make a pint, how many glasses of lemonade are there in three gallons?

4. Henry bought a peck of chestnuts for $1.25, roasted them and sold them for 10 cents a bag. How much did he make if he got four bags out of a quart?

5. The children of the Fairview School bought ribbon and cut it into pieces 4½ inches long to make badges to wear at the school carnival. How many yards did it take to make 180 pieces?

Practical problems involving the addition, subtraction, multiplication, and division of denominate numbers are not frequently found. Addition and subtraction occur somewhat more frequently than do multiplication and division and should be presented to the extent of a few simple examples. A lesson on such problems as the following will help the pupil to see the nature of the processes and to understand them when they are met.

1. Joe raised four turkeys and sold them just before Thanksgiving. They weighed 9 lb. 3 oz., 11 lb. 7 oz., 10 lb. 8 oz., and 11 lb. 3 oz. How much did they all weigh?

2. In a running broad jump, Jack jumped 14 ft. 9 in. and Bill jumped 12 ft. 11 in. How much farther did Jack jump than Bill?

3. Mrs. Richards bought two pieces of gingham at a remnant sale. The first piece contained 2 yd. 14 in. and the second piece, 1 yd. 28 in. How much was there in both pieces?

4. If it takes 2 hr. 55 min. for a train to run from Cleveland to Columbus and 2 hr. 45 min. to run from Columbus to Cincinnati, how long does it take to run from Cleveland to Cincinnati?

When denominate numbers occur in problems, they are frequently expressed as fractions or mixed numbers. Then in adding, subtracting, multiplying, or dividing, pupils are performing operations with which they are already familiar.

Teaching the operations with denominate numbers. The procedure of adding or subtracting denominate numbers is simple and straightforward. The principle

difficulty lies in carrying or borrowing. In the example shown, for instance, pupils tend to write 6 ft. 5 in., instead of 6 ft. 3 in., as the sum and to make a similar error whenever the sum of the first column is 10 or more. They carry 10 instead of carrying 12. Of course, this is a perfectly natural mistake since, heretofore, the carry number from the first column has been 10. It is important that the proper procedure be emphasized when the illustrative examples are explained. If pupils are closely supervised in their practice exercises, early errors may be detected and corrected.

ft.	in.
3	7
2	8
6	3

The teacher may solve on the blackboard and explain the first of the four problems which have been stated, as follows: "Write the numbers for pounds under *lb.* and the numbers for ounces under *oz.* Now, add the ounces first, getting 21 ounces. Remember that there are 16 ounces in a pound. Then, 21 ounces would be equal to 1 pound and 5 ounces. Write the 5 under *oz.* and carry 1 pound to the pounds column. Then, the four turkeys weighed 42 pounds and 5 ounces."

lb.	oz.
9	3
11	7
10	8
11	3
42	5

In a similar manner, a subtraction example may be worked out. We may illustrate with the second problem. We write the numbers which represent the distance Bill jumped beneath the numbers which represent the distance Jack jumped. We can not subtract 11 inches from 9 inches so we borrow a foot, or 12 inches. Since 9 inches plus 12 inches

ft.	in.
14	9
12	11
1	10

make 21 inches, we subtract 11 from 21. Then, we subtract 12 from 13. So, Jack jumped 1 foot and 10 inches farther than Bill.

Learn measures through use. If the pupils are to know and understand weights and measures, they must use them. They should have actual experiences with the measures studied. The mere memorizing of tables will be of little value. It is not uncommon to find pupils who can recite glibly the fact that there are 12 inches in a foot, 3 feet in a yard, 5½ yards in a rod, and 320 rods in a mile, but who have only the vaguest notion of the actual length of a foot, a yard, a rod, or a mile, particularly the last two. To know that there are 160 square rods in an acre may be worth while, but it is far more worth while to have had experiences with areas which are about an acre in extent and to be able to tell whether an observed plot of ground contains about an acre or much more or much less.

The ceiling of the author's classroom is about 12 feet high. On several occasions, teachers and prospective teachers who were students in the author's classes have been asked to estimate the height of this ceiling. Their estimates have ranged from 8 feet to 30 feet. The campus on which the building containing this classroom is located is about 10 acres in extent. Estimates of the area by students who know it well range from 2 acres to 30 acres. Klapper tells of having sixth-grade pupils estimate the height of an ordinary ceiling and getting answers ranging from 10 feet to 60 feet with only about 10 per cent approximately correct. He secured answers ranging from 35 feet to 200 feet from

college juniors on the height of a four-story building with 20 per cent approximately correct.[4]

We must expect this sort of thing as long as the pupil's experience with measures consists largely of memorizing tables and solving book problems. Textbook problems should follow many meaningful experiences with the units of measure concerned. Some of these experiences will be common to all the pupils but others will vary with the environment in which the pupils live. The classroom assignments should vary accordingly. Thus, the urban pupil may be interested in the fact that in his city 10 blocks make a mile while the rural child may be interested in the fact that a *hand,* used in measuring the heights of horses, equals 4 inches.

The child's experiences with measures will begin in the primary grades and last through all the years which he spends in the elementary and secondary schools. He will gradually extend his knowledge of measures and his skill in their use as need for them arises. Much that he learns will come about incidentally in connection with, or as a by-product of, other activities. It is neither necessary nor desirable that all of a table be learned in one lesson or in a series of consecutive lessons. Thus, the pupil in the first grade learns to tell the day of the week and the day of the month. Later, he learns that there are 7 days in a week, that this month contains 30 days and he learns to tell time by the clock. Still later, he learns that there are 60 minutes in an hour, 24 hours in a day and usually a little over 4 weeks in a month. Much later he may discover that there are 60 seconds

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[4] Klapper, Paul. *The Teaching of Arithmetic.* New York: D. Appleton-Century Company, 1934, p. 411.

in a minute and 100 years in a century. Probably, he will never meet the fact that there are 10 years in a decade while he is in the elementary school.

Thus, work with denominate numbers moves out of the long occupied status in which instruction was based on the drill theory and into a new status in which the meaning theory of instruction predominates.

QUESTIONS AND REVIEW EXERCISES

1. What are the two major systems of weights and measures in use in the present civilized world?

2. How old is the metric system? In what country was this system developed? How extensively have various governments adopted it as the official system? Has it been officially adopted by the United States? Is its use permitted in the United States?

3. What advantages are gained through the use of the metric system? How do you account for the reluctance of the people in this country to use it?

4. Is our system of weights and measures essentially the same as that used in Great Britain? With what differences between the two systems are you acquainted?

5. Why should early efforts at measurement make such extensive use of parts of the human body? What disadvantages are there in such measurements? It is thought that we owe our decimal number system to parts of the body also. What is the reason for this point of view?

6. Do you know any merely descriptive measures of distance, capacity, or time other than those given in this chapter?

7. Where in our system of weights and measures do you see evidences of a duodecimal number system? Of a sexagesimal number system? Of binary subdivision?

8. How early in the grades should pupils begin the study

of denominate numbers? What should be the nature of the earliest instruction? When should the study of this subject be completed?

9. When should tables be memorized in serial order?

10. Describe briefly the work in denominate numbers which has been recommended for the first three grades?

11. What work in denominate numbers is recommended for the intermediate grades?

12. What advantage is there in bringing into the class discussion miscellaneous bits of information about measures and measuring practices?

13. What was the supposed advantage of the long exercises in reduction which were formerly found in arithmetic textbooks? Why have such exercises been eliminated?

14. What was the disciplinary point of view in arithmetic teaching?

15. What is the major difficulty which pupils experience in learning to add compound denominate numbers? In learning to subtract such numbers? How can these difficulties be avoided?

16. Should teachers be able to estimate fairly accurately distances in yards and rods, areas in acres, and weights in pounds? Why? How can skill in making such estimates be developed?

17. Should pupils learn to make such estimates as those indicated in Exercise 16? What type of experience is necessary if pupils are to develop such skills?

18. Should the course of study in weights and measures be the same for all of the pupils in a given grade in a state? Why?

19. To what extent can weights and measures be learned through the use of the theory of incidental instruction?

20. To what extent should the teaching of measures be based on the meaning theory? How is this subject taught if instruction is based on the drill theory?

CHAPTER TEST

Read each statement and decide whether it is true or false. A scoring key will be found on page 534.

1. The metric system of weights and measures was adopted first by the French government.
2. The metric system of measures is a decimal system.
3. The English system of measures is a decimal system.
4. The British imperial gallon is larger than the gallon used in the United States.
5. The British imperial bushel is larger than the bushel used in the United States.
6. The total arm spread was once called the cubit.
7. The sexagesimal number system has a base of 6.
8. The English system of measures is based largely on the duodecimal number system.
9. The study of denominate numbers should be begun in the first grade.
10. The study of denominate numbers should be completed in the sixth grade.
11. The words *inch* and *ounce* have a common origin.
12. When pupils begin to study measures of length, they should memorize the table for such measures.
13. The first item to be learned in measures of time is the fact that there are 60 minutes in an hour.
14. The course of study in measures should be the same for rural pupils as for urban pupils.
15. Teachers should be able to estimate small areas in acres rather accurately.
16. In recent years, there has been a decrease in the amount of work in reduction of denominate numbers which is included in textbooks.
17. Multiplication of compound denominate numbers occurs more frequently in ordinary affairs than does subtraction of compound denominate numbers.

18-30. For each item in the first column, select an item in the second column which matches it.

18.	Farthing	(*a*)	2150.42 cubic inches
19.	Winchester gallon	(*b*)	Base of 60
20.	Imperial gallon	(*c*)	16½
21.	Winchester bushel	(*d*)	231 cubic inches
22.	Sexagesimal	(*e*)	32 quarts
23.	Duodecimal	(*f*)	Base of 10
24.	Hand	(*g*)	5280
25.	Feet in a rod	(*h*)	An English measure of money
26.	Crate of berries	(*i*)	1728
27.	Bushel	(*j*)	4 inches
28.	Feet in a mile	(*k*)	Metric measure of capacity
29.	Ordinary ton	(*l*)	277.274 cubic inches
30.	Liter	(*m*)	2240 pounds
		(*n*)	Base of 12
		(*o*)	24 quarts
		(*p*)	2218.192 cubic inches
		(*q*)	2000 pounds
		(*r*)	Quart

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12. Sanford, Vera. *A Short History of Mathematics.* Boston: Houghton Mifflin Company, 1930, 402 pp. Interesting historical material on weights and measures is presented in Chapter XIII, pages 353-377.

13. Schaeffer, Grace E. "An Informational Unit on Time." *Elementary School Journal:* XXXVIII: 114-117, October, 1937. Describes the collection of data for time measurement as a unit for the fourth grade.

14. Wheat, Harry Grove. *The Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937, 591 pp. A well written and stimulating discussion of measures and weights will be found in Chapter XX, pages 471-498.

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CHAPTER 10

THE ELEMENTS OF MENSURATION

Mensuration in the intermediate grades. There is considerable variation in courses of study as to the amount of attention which should be given to elementary work in mensuration in the intermediate grades. Some provide for none; others begin the subject early and carry the program to the point where pupils do many constructions to scale, learn to find areas of many plane figures, calculate volumes, find perimeters, do related work in graphs, and solve involved problems in which principles and facts learned in mensuration are employed.

Mensuration can easily be made too difficult for pupils in these grades. The difficulty may inhere in the fact that pupils can see no real uses for what they learn, it may be due to the abstract nature of the examples and the problems, or it may be due to the teaching method employed, a method which is too often based on the drill theory of instruction. On the other hand, if the materials of instruction are selected in the light of pupil interest and pupil experience and if the method is one in which the meaning theory is incorporated, the elements of mensuration will not be too difficult for pupils in the intermediate grades. It may be easy enough for pupils to attain to a reasonable degree of success in it and it may also be interesting and worth while.

The work in measurement which is done in the primary grades and continued in the intermediate grades

leads quite naturally to the elements of mensuration. Measures of distance may include lengths and widths of rectangular figures, for example. Then it is but a short step to the measurement of perimeters and not a very long step to the measurement of areas.

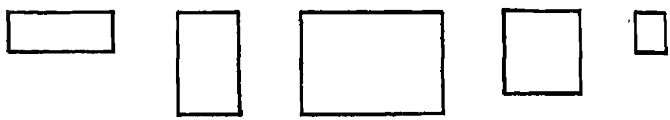
Pupils in the intermediate grades may study the rectangle, its special case, the square, the triangle, and they may have some experience with the circle. They will learn to recognize each of these figures by its characteristic shape. They will learn that a rectangle may be a square but that a square is always a rectangle. They will have experiences in finding perimeters. And they will learn to find areas of rectangles and triangles. Some would say with justification, that the triangle should be postponed until the junior high school years but some attention may be given to the right triangle in the intermediate grades since two right triangles are formed by the diagonal of a rectangle.

Defining plane figures. It is well for the more mature student to learn as a summary of his experiences with plane figures that a quadrilateral is a plane figure bounded by four straight lines, that a parallelogram is a quadrilateral whose opposite sides are equal and parallel, that a rectangle is a parallelogram having four right angles, and that a square is a rectangle whose sides are equal. But for the beginner, such definitions are more of a hindrance than a help. If the pupil in the intermediate grades can identify a square, a rectangle, and a triangle, as well as a circle, it matters little whether or not he knows or remembers formal definitions. If given at all, the definitions should come *after* a

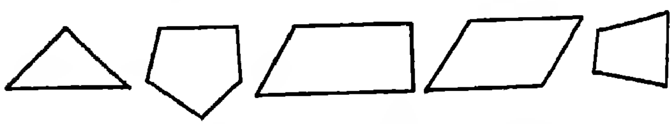
series of interesting, meaningful experiences with the things defined.

Early explanations should be made in terms of actual examples. These examples may be objects in the schoolroom and elsewhere and they may be drawings which are placed on the blackboard and on sheets of paper. These illustrations should be numerous and varied. When drawings are used, the development may proceed in some such manner as the following:

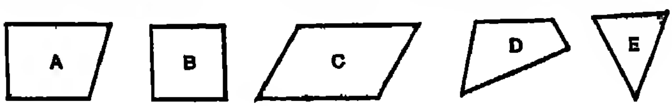
We call figures like these *rectangles.*

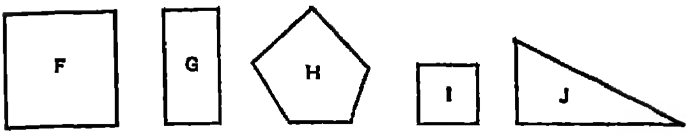


You see, rectangles always have four sides and all four corners are square. Figures like those in the group below are not rectangles because some of them do not have four sides and in none of them are all of the corners square. Can you tell which corners are not square? How many sides are there in each of these figures?



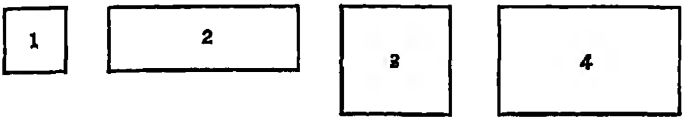
Look at the figures in the next two groups and tell which are rectangles. Is figure A a rectangle? How do you know? Is B a rectangle? How can you tell? Answer the same questions for the rest of the figures.





Look at the four rectangles in the next group of figures. Two of them are *squares*. Can you tell which two are squares?

You see, if all four sides of a rectangle have the same length, the rectangle is a square. Do the four sides of number 1 have the same length? Number 2? Is number

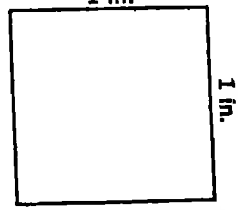


3 a square? How can you tell? Is number 4 a square? Why? Look again at the 10 figures which are lettered A to J. Are any of them squares?

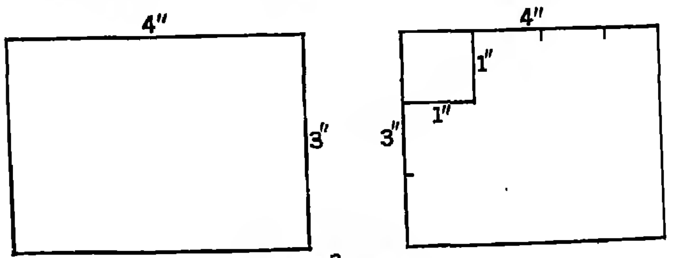
After a few lessons with rectangles the term, *right angle*, may be introduced and used in the place of *square corner*. No formal definition of *right angle* is necessary at this time. Nor is it necessary to use and define the term, *angle*, before *right angle*. The pupil simply learns that when two straight lines meet so as to form a square corner, they form a right angle. This is learned from observation; it is not formally memorized. Then, it is seen that a rectangle has four right angles.

Finding the area of a rectangle. The first step toward finding areas of plane figures is for the pupil to gain an understanding of what is meant by 1 square inch, 1 square foot, etc., as a unit of measurement. The pupil

learns that if a square, each side of which is 1 inch in length, is drawn, the area of the square is 1 square inch. Then, if a rectangle four inches long and three inches wide is drawn, our problem is to find the number of square inches in the area of this rectangle.



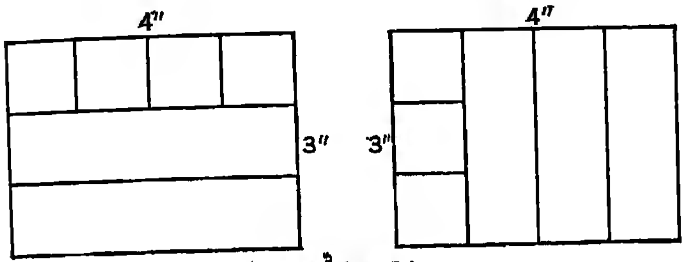
First, we draw the rectangle four inches long and three inches wide. Then, we draw 1 square inch in a corner of the rectangle, as shown, and place marks to divide the remainder of each of two adjacent sides into one-inch lengths. The pupils see now that we are to find how many of these square inches there would be in the entire area of the rectangle.



Scale $\frac{3}{8}$ in. = 1 in.

The next step is to lay off a strip of square inches, that is, a strip 1 inch wide which is then divided into square inches. These strips may run either lengthwise or crosswise. Two figures may be drawn so as to make them run

both ways, as shown. Now, the pupils see that there are 4 square inches in a strip, if the strips run lengthwise. Since there are 3 strips, there would be 3×4 square inches, or 12 square inches, in the figure. Again, if the strips run crosswise, they see that there are 3 square inches in a strip and 4 strips and in the entire rectangle, 4×3 square inches, or 12 square inches.



Scale $\frac{3}{8}$ in.=1 in.

Of course, each of the strips may be cut into square inches and the pupil may simply count the number of square inches in the entire figure to determine that there are 12. It seems better, however, to use the plan which has been sketched. Perhaps, we may go a step farther and divide the entire rectangle into one-inch squares if the pupils, particularly the duller pupils, are unable to see what the area would be from the figures which have been drawn, but this should be done only as a last resort.

At any rate, pupils should soon see that the number of square inches in a strip is equal to the number of inches in the length (or width), that the number of strips is equal to the number of inches in the width (or length), and that the number of square inches in the

entire figure is equal to the number of square inches in one strip multiplied by the number of strips.

Two or three examples should be worked out in the objective manner which has been illustrated. In these, various units of measure should be used and various dimensions should be employed. Gradually, the pupil generalizes to the effect that *length times width equals area*. There should be frequent repetition of such statements as the following:

Length in inches times width in inches	=	area in square inches
Length in feet times width in feet	=	area in square feet
Length in yards times width in yards	=	area in square yards
Length in rods times width in rods	=	area in square rods
Length in miles times width in miles	=	area in square miles

From such statements as the preceding and those which follow, the formula will gradually evolve.

The area	equals the	length	times the	width.
Area	equals	length	times	width
Area	=	length	$\times$	width
A	=	l	$\times$	w

$$A = lw$$

The formula, $A = lw$, thus summarizes in a brief, concise manner what the pupil has learned. Coming as the culmination of a series of interesting concrete experiences, it is meaningful and is likely to be retained. If it is forgotten, the pupil probably can recall it by constructing rectangles and reproducing some of the experiences by which the formula was originally obtained.

Although the pupil's first concept of the square inch, the square foot, etc., is that of a square each side of which is 1 inch, 1 foot, etc., in length, he must soon

learn that a figure may have an area of 1 square inch, 1 square foot, etc., although it is not a square. Several examples of this should be illustrated objectively. Thus, a figure 2 feet long and ½ foot wide may be drawn and the pupil may discover that the area is 1 square foot by cutting the figure crosswise into two equal parts and fitting the parts together to make a figure one foot square. Again, he sees that the area may be found by using the formula, $A = lw$.

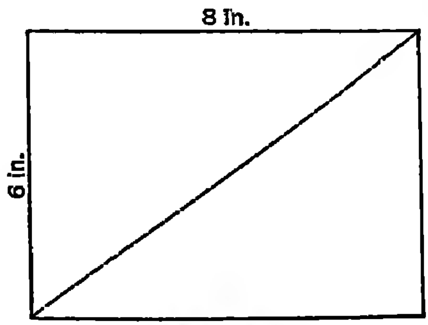
Such expressions as, "3 in. $\times$ 4 in. = 12 sq. in." or "3 $\times$ 4 $\times$ 1 sq. in. = 12 sq. in." should be avoided, the former because it is meaningless and the latter because it is pedantic. We have discussed briefly the subject of abstract and concrete numbers (pages 166-170). Pupils should not be disturbed by rules for abstract and concrete numbers when learning the elements of mensuration but the teacher should set good models for them to follow. Inches times inches do not give square inches, or anything else, but *length* in inches *times width* in inches *does give area* in square inches.

The older method was to start with the formula, or its equivalent in the form of a verbal statement of a rule. After memorizing a definition of a rectangle, the pupil was confronted with some such rule as the following: The area of a rectangle is the number of units in the length multiplied by the number of units in the width times the unit of measure. To all but the most gifted such a rule was meaningless and probably did more harm than good.

Pupils frequently make errors when the dimensions are expressed in different units of measure. From the beginning, emphasis should be placed upon the neces-

sity of expressing both dimensions in the same unit of measure. This is done by repeating such statements as, "Length in feet times width in feet equals area in square feet," by solving problems in which the dimensions are expressed in different units, and by using diagrams drawn accurately to scale with the dimensions written in their appropriate places.

Finding the area of a triangle. In this discussion, we shall refer to the right triangle only. The pupils learn to recognize a plane figure bounded by three straight lines as a triangle. They can easily see that if a diagonal is drawn in a rectangle, two triangles are formed. They also see that these triangles are equal. It is well to construct such a figure carefully on paper, cut out the figure, cut it into two triangles by cutting along the diagonal, and show that the two triangles are equal by superimposing one on the other. Each pupil may do this for himself and the rectangles with which the various pupils work need not have the same dimensions.



Scale 1 In.=4 In.

Now, the pupils have learned that the area of a rectangle is found by the formula, $A = lw$. Then, the

area of either of the two triangles would be given by the formula,

$$A = \frac{1}{2}\,lw, \text{ or } A = \frac{lw.}{2}$$

In the figure on page 440, the area of the rectangle is 48 square inches and the area of the triangle is one-half of 48 square inches, or 24 square inches.

Practice in finding areas. The only practice which many pupils receive in finding areas of rectangles, triangles, and other plane figures which they study comes from the solution of textbook problems, such as, "Find the area of a rectangle whose base is 12 inches and whose altitude is 4 inches," or, "A triangle has a base of 8 feet and an altitude of 6 feet. Find the area." Practice of this kind is good enough as far as it goes but it is very inadequate if pupils are to be enabled to find areas of real rectangles and triangles at home and elsewhere in out-of-school activities. A pupil who has become proficient in solving textbook problems such as those just stated may be utterly unable to find the area of the floor of a room in his home. He may even be unable to find the area of a figure when the drawing is presented to him with the dimensions properly written in, because of his inability to tell which is the length and which is the width or which is the base and which is the altitude.

The remedy lies in providing problems with a more realistic setting and in correlating the finding of areas with actual work in measuring. The class may begin by measuring and finding areas of surfaces of objects close at hand such as a page of a book, a sheet of paper, the top of a desk, the top of a table, a pane of glass, a

section of blackboard, the floor of the classroom, etc. Later, garden plots, a section of playground, a vacant lot, the floor of a room at home, etc., may be measured and the areas found. Practical problems pertaining to the buying of paint and varnish may be used to give the exercises a real setting and to make them more interesting.

Good practice may be provided by cutting heavy cardboard into rectangles and triangles of various dimensions and letting pupils measure the pieces and compute the areas. The first pieces should be cut very carefully so as to be just 6 inches, 9 inches, 4 inches, etc., in length or width. Later, pieces may be cut without reference to any particular number of inches and practice provided in measuring to the nearest inch, the nearest half-inch, the nearest quarter-inch, or the nearest eighth-inch. This will give practice in careful measurement and in the multiplication of mixed numbers. The answers which the various pupils obtain naturally will differ somewhat but the teacher need not be concerned about this if the differences are not great for expert surveyors do not get exactly the same result in finding the area of a tract of land. In measuring triangular pieces, pupils should be permitted to use any convenient side as the base if they use as the altitude the perpendicular distance from that side to the opposite vertex. However, early work with triangles should be limited to right triangles, as has been suggested. When cardboard figures have been cut and measured and the areas have been computed by the teacher, they may be numbered and filed away with the data pertaining to them, for the use of later classes.

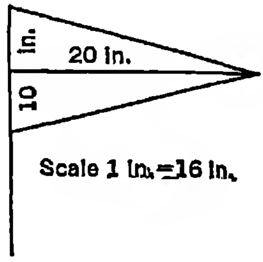
Some teachers object to the kind of practice material which has just been suggested. They complain that there are no certain answers with which the pupils can compare their results. Right here lies a significant merit of this kind of exercise. Practice and good measuring instruments enable teachers and pupils to increase the skill with which they measure and the accuracy of their measurements but both teachers and pupils should learn that there is no such thing as an exact measurement. The pupil's concern, both in and out of school, should be to make the most accurate measurements of which he is capable. He should learn early that errors will be made and that individual measurements will vary. One pupil registered his distress and his confusion by saying, when he was handed a cardboard triangle and was asked to find its area, "Mine ain't got no base or altitude."

The following problems are suggestive of the kinds which may be used. They are deliberately chosen so as to represent a rather wide variety of conditions. They should suggest other problems which may be better adapted to the needs and interests of the pupils in a particular class.

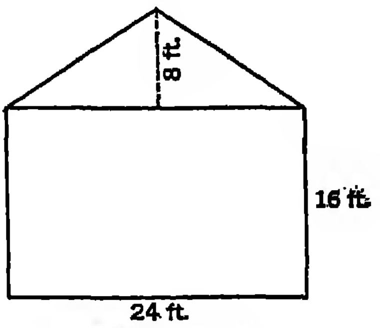
1. Measure a sheet of your tablet paper. What is the length? What is the width? What is the area?
2. Find the number of square inches in the top of your desk. The number of square feet. How do you change square inches to square feet?
3. A quart of varnish will cover 150 square feet. Would a quart be enough to varnish the tops of the desks of all the pupils in your room?
4. There should be at least 18 square feet of floor area

in a schoolroom for each pupil. How many square feet are there for each pupil in your room?

5. This figure represents a pennant made of a piece of red felt and a piece of black felt. How many square inches of each kind does it take to make a pennant?



6. Mr. Dennis is getting ready to paint his barn. This figure represents one end of the barn. How much paint will it take to cover this end if a quart will cover 125 square feet? How much will be needed to paint both ends?



Scale 1 in.=16 ft.

7. If Mr. Dennis' barn is 36 feet long, how much paint will be needed to paint the two sides?

8. Which is larger, a 3-inch square or 3 square inches? How much larger? How many times as large?

9. Ralph, who lives on a farm, wanted to find how many acres there were in his father's corn field. He found that the corn was planted in rows both ways across the field. The rows were three feet apart. He counted the rows and found that there were 284 one way and 192 the other. The field was how many rods long? How many rods wide? How many acres in the field?

10. The number of square feet of window glass in a schoolroom should be about one-fifth the number of square feet in the floor. Is this true of your schoolroom?

11. A rug 6 feet long and $4\frac{1}{2}$ feet wide costs \$4.50 and a rug 12 feet long and 9 feet wide costs \$16.00. Which costs the more per square foot?

12. Dick is going to build a chicken yard back of the garage. It will be 18 feet long. If he allows 6 square feet for each chicken, how wide should the yard be to make room for 40 chickens?

Finding perimeters. While the rectangle and the triangle are being studied and areas are being found, it is an interesting and valuable exercise to find perimeters also. Problems requiring the amount of fencing needed for a lot or a garden plot, border for articles made in sewing, material to frame a picture, etc., are easily found. Some of the problems assigned will be combination problems, requiring that both the area and the perimeter be found. In number 12 of the above list, for example, pupils may find the amount of fencing which Dick will need. This may be further complicated by the fact that the garage wall forms one side of the rectangle.

Drawing to scale. In the first problems with plane figures which are used in class and which the pupils solve, the dimensions should be small numbers of inches so that the drawings may actually have the dimensions

which the problems give. Thus, we may talk about a rectangle 5 inches long and 3 inches wide and construct a figure just 5 inches long and 3 inches wide. Larger rectangles may be drawn on the blackboard. But in time, problems will be encountered which give dimensions in yards, rods, and even miles. It is then necessary that the diagrams be constructed on a reduced scale. Pupils should learn to draw to scale.

In teaching pupils to draw to scale, we should use a series of exercises which are of gradually increasing difficulty. For example, it is much easier for the beginner to represent by a scaled drawing a rectangle 24 feet long and 16 feet wide than a rectangle 28 feet long and 15 feet wide since the dimensions of the former contain common integral factors while the dimensions of the latter do not. In the former, an inch can represent 8 feet or 4 feet and the drawing will be of convenient size. In the latter, if the pupil lets one inch represent 3 feet, 5 feet, 6 feet, or 7 feet, fractions are obtained which are not readily measured with an ordinary rule. Probably, the best scale in this case is that in which one inch represents 4 feet for this will make the drawing 7 inches long and 3¾ inches wide. Skill in choosing a desirable scale grows slowly and gradually with experience but for its best development, the exercises should be planned and graded with care.

Pupils learn that a scaled drawing differs in size from that which it represents but that it has the same shape. The size may be changed to suit our convenience but the shape remains the same. It is well to have a garden, a room floor, or other rectangle represented by drawings having two or three different scales in order that pupils

may see that the size of the drawing is not very important but that the shape must not be changed. Floor plans of houses are easily found in magazines and may be examined by the pupils and the scales noted. Local large-scale maps of a city, showing the streets and principal points of interest, and of a rural neighborhood, showing highways, railroads, etc., can be obtained and studied. Maps in geography texts can be examined and the scales compared and discussed. Pupils learn that they can not tell about the comparative sizes of countries by comparing maps unless the scales of these maps are taken into account.

Drawing to scale is a valuable type of training. It not only helps the pupils in the visualization of the conditions of a problem in mensuration but also it forms the basis for later construction work in geometry and for the graphic representation of statistical data.

Finding the volume of a rectangular solid. When pupils begin their work with three-dimensional figures, they should have concrete materials to illustrate the observations made since they find it difficult to visualize the details of three-dimensional space even when good drawings are made. It is well to have a supply of one-inch cubes and a box each of whose dimensions is an integral number of inches. By actually placing the cubical blocks in the box, the pupils see that the number of inches in the length of the box tells the number of blocks required for a row, that the number of inches in the width of the box indicates the number of rows of blocks in a layer, that the product of these two numbers gives the number of blocks in a layer, that the number of inches in the height, or depth, of the box tells the

number of layers of blocks, and that the product of the length, the width, and the height gives the cubic contents of the box, or its volume.

Of course, the first concept to be developed is that of the cubic inch, cubic foot, or whatever is to be used as the unit of measure. Again, the concrete representation, in the form of the cubical block, is to be recommended. The pupil learns that a block which is an inch long, an inch wide, and an inch high, contains one cubic inch. This should be associated with the square inch. Later, the pupil broadens his concept of the meaning of a cubic inch by learning that a block may contain a cubic inch even though the dimensions are not equal. For example, a block, two inches long, one inch wide, and one-half inch thick contains one cubic inch.

From concrete examples, the formula for the volume of a rectangular solid is gradually evolved in a manner similar to that used in developing the formula for the area of a rectangle. The pupil learns that the dimensions must all be in the same unit of measure; that length in inches times width in inches times height in inches equals volume in cubic inches; that length in feet times width in feet times height in feet equals volume in cubic feet; etc.; and then, in general, that volume equals length times width times height; or, $V = lwh$.

The number of cubic inches in a cubic foot and the number of cubic feet in a cubic yard should be worked out by the pupils from their knowledge of volumes of rectangular solids just as the number of square inches in a square foot and the number of square feet in a square yard should be deduced from their knowledge of areas of rectangles. Thus, each pupil discovers for himself

that there are 144 square inches in a square foot, 9 square feet in a square yard, 1,728 cubic inches in a cubic foot, and 27 cubic feet in a cubic yard. It is much better to develop these facts from known facts in linear measure than to memorize them as separate and independent facts. If facts which have merely been memorized are forgotten, they are lost; but if facts which have been derived from other known facts are forgotten, they can be derived again.

QUESTIONS AND REVIEW EXERCISES

1. To what extent can the seventh-grade teacher of arithmetic or junior high school mathematics assume that the pupils have already been instructed in the elements of mensuration?

2. Some teachers contend that mensuration is too difficult for sixth-grade pupils. Is this true? What reasons can you suggest for this contention?

3. How is the measurement of area related to the measurement of distance? How is the measurement of volume related to the measurement of area?

4. Should the parallelogram have been included in the list of plane figures treated in this chapter? Why?

5. How early in his study of the rectangle and the triangle should a pupil learn a formal definition for each of these figures?

6. What is the difference between a square and a rectangle? Between a rectangle and a parallelogram? Between a parallelogram and a quadrilateral? Is a square a quadrilateral?

7. How early should the term *right angle* be introduced? What does a pupil call a right angle before he learns this term? Should the term *angle* be used and defined before the term *right angle* is introduced?

8. Outline the development suggested for leading the pupil to see that the area of a rectangle is the product of the length and the width.

9. Thorndike objects to this development on the ground that it leads the pupil to believe that a rectangle must be made up of a whole number of individual one-inch squares and that he can not see the application of the formula to such a rectangle as one $3\frac{1}{2}$ inches long and $2\frac{1}{4}$ inches wide. What do you think of this objection?

10. Why should one not say that inches times inches gives square inches or that inches times inches times inches gives cubic inches? Why is it not recommended that one write "$6 \times 5 \times 1$ sq. in." in finding the area of a rectangle 6 inches long and 5 inches wide?

11. Why is it better to have the pupils develop the formula, $A = lw$, from objective experience with drawings than to have them memorize the formula at the beginning of their work with rectangles?

12. Why is it that pupils who can solve textbook problems on areas of plane figures sometimes can not find the areas of such surfaces in out-of-school situations?

13. If two pupils measure and find the area of the same triangle, will they obtain the same answer? Why?

14. If cardboard triangles are supplied to the pupils for practice in finding areas, should the bases and altitudes be labeled? Why?

15. Can the dimensions of a plane figure be determined exactly? Can the exact area be found?

16. If the length and width of a rectangle are each determined to the nearest inch and are found to be 10 inches and 8 inches respectively, what is the maximum possible error in the area obtained?

17. Suggest a variety of practical applications for work in finding perimeters of rectangles.

18. Should sixth-grade pupils learn to draw to scale? Do they have need for an understanding of scale drawing aside from that which arises in connection with their work in arithmetic?

19. Why is it advisable to use concrete materials instead of drawings in teaching pupils to find volumes of rectangular solids?

20. Should a pupil learn from a table the number of square inches in a square foot and the number of cubic inches in a cubic foot or should he discover these facts for himself? Why? How can he be expected to discover such facts for himself?

CHAPTER TEST

Determine whether each statement is true or false. A scoring key will be found on page 534.

1. The recommended list of plane figures for study in the intermediate grades includes the parallelogram.

2. When the pupil begins the study of the triangle, he should memorize a definition of the triangle.

3. All rectangles are squares.

4. All squares are rectangles.

5. All rectangles are parallelograms.

6. To find the area of a rectangle 8 inches long and 6 inches wide, the pupil should be taught to multiply 1 square inch by 6×8.

7. Feet times feet gives square feet.

8. The only type of triangle treated in this chapter is the right triangle.

9. Pupils can not be expected to make exact measurements.

10. If two persons find the area of the floor of a room, they usually get the same answer.

11. If cardboard triangles are supplied to pupils for

practice in finding areas, the base and altitude should be labeled.

12. The perimeter of a plane figure is the distance around it.

13. Pupils in the intermediate grades should learn to draw to scale.

14. A scaled drawing should have the same shape as the original which it represents.

15. When pupils are learning to find the volume of a rectangular solid, the meaning of the steps should be represented by drawings.

16. Pupils should discover for themselves the number of cubic inches in a cubic foot.

17-26. For each item in the first column, select an item in the second column which matches it.

17. A rectangle
18. A right angle
19. Area of rectangle
20. Area of triangle
21. Perimeter
22. Quadrilateral
23. A triangle
24. A square
25. A cubic foot
26. Volume of rectangular solid

(*a*) $A = lw$
(*b*) The distance around a figure
(*c*) A plane figure bounded by three straight lines
(*d*) An angle of 90 degrees
(*e*) A rectangle having equal sides
(*f*) A parallelogram having right angles
(*g*) $A = \frac{1}{2} lw$
(*h*) The diagonal
(*i*) 2150.42 cubic inches
(*j*) A plane figure bounded by four straight lines
(*k*) 1728 cubic inches
(*l*) 144 square inches
(*m*) An acute angle
(*n*) 231 cubic inches
(*o*) $V = lwh$

SELECTED REFERENCES

1. Glazier, Harriet E. *Arithmetic for Teachers.* New York: McGraw-Hill Book Company, 1932, 291 pp. Mensuration and intuitive geometry are treated in Chapter VIII, the treatment going beyond that required for the intermediate grades.

2. Klapper, Paul. *The Teaching of Arithmetic.* New York: D. Appleton-Century Company, 1934, 525 pp. Very worthwhile suggestions are offered on pages 416-424.

3. Lindquist, Theodore. *Modern Arithmetic Methods and Problems.* Chicago: Scott, Foresman and Company, 1917, 300 pp. Considerable informational material of value to the teacher will be found in Chapter XXV, pages 256-271.

4. Roantree, William F. and Taylor, Mary S. *An Arithmetic for Teachers.* New York: The Macmillan Company, 1932, 523 pp. Chapter IX, pages 304-340, is entitled "Geometric Measurements." It contains valuable material for the teacher although much of it has to do with the course beyond the sixth grade.

5. Taylor, E. H. *Arithmetic for Teacher-Training Classes.* New York: Henry Holt and Company, 1937, 432 pp. The teacher whose scholarship in areas and volumes is inadequate should study the material and solve the exercises of Chapter XIX, pages 369-390.

CHAPTER 11

SOLVING PROBLEMS

Why should problem solving be emphasized? It is the chief purpose of arithmetic instruction to teach pupils to solve problems. Frequently, it has been emphasized that as pupils learn the basic operations with numbers, they should apply their skills in the solution of problems.[1] In the preceding chapter, many new types of examples have been introduced through the medium of problems. These operations are taught in order that the pupils may be able to solve the problems which they encounter in their school days and in their after-school experiences. Skills in the fundamental operations are not ends in themselves; they are the means to an end. The end is the ability to solve problems.

Distinction between a problem and an example. In this volume and in the one which precedes it, the word "problem" has been used consistently to refer to an arithmetical situation in which the operation to be performed is not indicated but must be decided upon by the pupil. A problem is a verbal statement, followed by a question or a demand; hence, problems are often called "verbal problems" or "word problems," although these latter terms are not often used except by those who call all arithmetic exercises "problems." Examples, on the other hand, are exercises in arithmetic which

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[1] Morton, Robert Lee. *Teaching Arithmetic in the Elementary School, Volume I, Primary Grades.* New York: Silver Burdett Company, 1937, pp. 107-110, 202, 246-249, 273-274, and 315. See also pages 346-369.

are accompanied by instructions in the form of verbal directions or signs which indicate to the pupil the operations which he is to perform. Examples give the pupil practice in and test his skill in the performance of the fundamental operations with numbers. Problems require the pupil to appraise situations and to decide upon the operation or operations which are to be performed and the order in which they are to be performed if there is more than one operation.

This distinction between a problem and an example is an arbitrary one, to be sure. It is an arbitrary one because some problems are so simple and the solutions are so obvious that they are hardly more than examples to the pupils who solve them while examples in new and involved operations may be real "problems" to pupils who are just learning to solve them. But the distinction is a convenient one for teachers to make as they organize their lesson plans and provide pupils with exercises for practice.

Factors conditioning success in problem solving. The author has given a battery of tests to 300 pupils in grades five, six, seven, and eight in a small city school system to determine the extent to which problem solving ability is related to each of several other factors or abilities. For each of the 300 pupils, information was obtained on the following points:

1. The ability to solve arithmetic problems.
2. Verbal intelligence.
3. Non-verbal intelligence.
4. Skill in the fundamental operations of arithmetic.
5. Comprehension in silent reading.
6. Rate in silent reading.

7. Regularity in school attendance.
8. Age in months.

The relation of problem solving ability to each of the other factors or abilities was determined by the method of correlation. This method yields a *correlation coefficient,* a decimal fraction which tells how closely two sets of scores or other variables are related. The correlation coefficients which tell how closely problem solving ability is related to each of the other seven factors are given in Table 8.

TABLE 8. CORRELATIONS WITH ABILITY IN PROBLEM SOLVING

Factor or Ability	Correlation with Ability in Problem Solving
Verbal intelligence	.78
Non-verbal intelligence	.52
Skill in the fundamental operations	.70
Reading comprehension	.61
Reading rate	.23
School attendance	.11
Age in months	.34

The larger the correlation coefficient, the closer the relationship. If the correlation between two sets of scores were perfect, that is, if the pupils all had the same rank in both tests, the correlation coefficient would be 1.00. Of the seven correlation coefficients given in Table 8, some are much higher than others. Apparently, the ability to solve arithmetic problems is rather closely related to intelligence, to skill in the fundamental opera-

tions, and to reading comprehension. This is what one would naturally expect. To solve problems as they appear in tests and textbooks, pupils must have brains; they must be competent in addition, subtraction, multiplication, and division; and they must be able to read with understanding. On the other hand, reading rate, regularity in school attendance, and age were not found to be very important factors. Again, this is what one would have expected. Reading rate surely is not so important as reading comprehension for one who is trying to grasp the conditions of a printed problem. What little correlation there is here probably is due to the fact that good readers usually read more rapidly than do poor readers. The low correlation with school attendance is due to the fact that none of the pupils had been absent excessively and that within narrow limits there is little relationship between attendance and school success. The correlation with age would have been much higher if all of the children in these schools had been tested, for the age range is considerably restricted in these four grades and there is some tendency for dull pupils to be retarded in grade progress and for bright pupils to be accelerated.

There is considerable evidence, according to psychologists, that what we call "intelligence" is largely an inherited trait and is not modifiable to a very great extent. That is, if a pupil is dull, there is not much which the teacher can do about it. This sounds like a fatalistic doctrine but it need not be so. The teacher can improve on current practices in making use of the intelligence which the pupils possess and in providing opportunities for their maximum growth and development. The

teacher can adapt the materials and methods of teaching to the intellectual level of the dull pupil but she can not cure his dullness. Stupid children can not solve complicated, difficult problems. No matter what the skill of the teacher or how nearly adequate the materials of instruction, there is a very definite limit to the difficulty of those arithmetic problems which pupils of a given intelligence level can solve.

But children of equal intelligence differ considerably in the success which they attain in problem solving. Other factors condition that success. We do not know what all of these are but our correlations tell us that two of them—and two very important ones—are (1) skill in the fundamental operations of arithmetic and (2) comprehension in silent reading. No teacher who would have her pupils accomplish all that they are capable of accomplishing in problem solving can afford to neglect these two important factors.

It is easy to see how success in problem solving must depend upon skill in the fundamental operations of addition, subtraction, multiplication, and division. However correct the pupil's reasoning may be, if he can not perform operations correctly after he has decided what they are, his answers must be wrong. We see here the fundamental operations in their proper rôle—tools which enable the pupil to carry out the solutions which he has devised.

Nor is it surprising that the ability to read printed paragraphs understandingly counts heavily in solving arithmetic problems. Before the pupil can begin to devise a solution for a problem, he must know what the conditions of the problem are and he must understand

clearly what is required. Ordinarily, he gets this information by reading a printed or written statement of the problem as given in a textbook or elsewhere. Skillful reading precedes the pupil's efforts to think out the steps of the solution. Accurate computation follows and permits him to put his thinking into full effect.

What has just been said applies particularly to the solution of problems in school. If the problems which a pupil solves are drawn largely from books and lists prepared and written by his teacher, reading and arithmetic skills are of almost equal importance. Each is indispensable. In actual life experiences, however, problems are not written out; they occur as integral parts of larger and more extensive experiences. What is demanded is apparent with no reading. Hence, skill in the fundamental operations is probably more important than skill in reading so far as the solution of such problems is concerned.

Not the least important of the findings of the study which has been reported very briefly is the fact that problem assignments should be varied to suit the greatly varying capacities of the children to be taught. What is suitable for average children is too easy for the gifted and too difficult for the dull. Providing a uniform assignment for all is likely to result in a loss of interest on the part of the brighter children because the tasks are not a fit challenge to their abilities and an equal or greater loss of interest on the part of the duller because they fail to understand what is required of them. Only by making the assignments conform to the differing capacities and interests of the pupils in the class can we hope to accomplish a maximum of benefit for all.

Summarizing the preceding paragraphs, we may say that the teacher who would get better results in problem solving may proceed as follows: (1) apply tests of attainment in problem solving, intelligence, reading, and fundamental arithmetic skills; (2) where deficiencies are found to exist, teach for better skills in the fundamental operations of arithmetic and in reading; (3) adapt problem assignments to individual differences among the pupils.

Further causes of difficulty in problem solving. Many plans for increasing the success of pupils in problem solving have been tried experimentally and have been written up in periodical literature and in monographs in recent years. One bibliography includes 85 references and, of course, no bibliography is complete.[2] The author has examined a vast amount of published material on this subject—thousands of pages—but has found few specific suggestions which can be relied upon to produce better results with pupils. Many of the suggestions are based upon experiments conducted with small numbers of pupils and some others are of the subjective or opinion type. Some of the results secured by different investigators fail to agree.

There seems to be rather general agreement that the points which have been mentioned are points of great importance. Intelligence is probably the most important single factor. But the teacher should remember that a dull pupil who can not solve the problems which are in the assignment for the grade in which he is located may well be able to solve these problems when he is

[2] Wilson, Guy M. "Bibliography—Written Problems in Arithmetic." *Education,* LIV: 480-483, April, 1934.

older and intellectually more mature. In the meantime, he should be given easier problems rather than be subjected continuously to the experience of failing what he undertakes to do.[3] Hence, the plea which has been made to adapt the problem assignments to the varying abilities found among the pupils.

The ability to perform the required computations is also of fundamental importance and, if the problems are book problems, it is quite necessary that the pupil be able to read them understandingly. To read problems successfully, the pupil must not only be able to get meaning from printed paragraphs but also he must be acquainted with the more or less technical terms which are used. One investigator found a pupil who was bewildered by the term, "hogshead."[4] There may be little justification for including "hogshead" in an intermediate grades problem but if it is there, it can be a real stumbling block for the pupil who does not know its meaning. A very elaborate study by Hydle and Clapp in which 350,000 problem solutions were analyzed showed a very consistent difference in favor of problems in which familiar terms were used as opposed to those in which unfamiliar terms were employed. For example, pupils who easily found costs in terms of units of United States money often were stopped by problems in which costs were expressed in terms of *kopecks*.[5]

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[3] Osburn, Worth J. *Corrective Arithmetic, Volume II.* Boston: Houghton Mifflin Company, 1929, p. 7.

[4] Gray, Olive. "Teaching Pupils To Read Arithmetic and Other Subject Matter." *Elementary School Journal,* XXVI: 607-618, April 1926.

[5] Hydle, L. L. and Clapp, Frank L. *Elements of Difficulty in the*

The authors of textbooks have been guilty of using words in arithmetic problems and in directions for solving problems which are not technical arithmetic terms but which are unfamiliar to most pupils for whom the problems were intended. Foran reports finding "estimate reasonable answers" in text material designed for pupils in the third grade.[6] He found one pupil who thought a reasonable answer would be a "cheap answer."[7] The author has found the following words in textbooks designed for use in grades five and six: *approximate, calculation, consumption, detect, efficiency, facility, horizontal, innumerable, installment, investment, maxim, obtainable, promote, redemption, transaction*. These words are in Thorndike's list of ten thousand, but are not sufficiently common to be in the first five thousand.[8] The following words, taken from the same books, are of such rare occurrence that they are not included in Thorndike's list of ten thousand: *athlete, boulevard, computation, compute, consecutive, dissimilar, divisibility, extreme, familiarize, frequency, intersect, organdie, parsnip, payroll, realization, salsify, systematic, tonnage, unclaimed, usable*. No effort has been made to compile a complete list of the words which are not included in Thorndike's list of ten thousand. Those given were found in a short time by leafing through a few pages of the books used. Although the

Interpretation of Concrete Problems in Arithmetic. Madison, Wisconsin: University of Wisconsin, 1927, p. 41.

[6] Foran, T. G. "The Reading of Problems in Arithmetic." *Catholic Educational Review*, XXXI: 601-612, December, 1933.

[7] *Ibid.*, p. 607.

[8] Thorndike, Edward Lee. *The Teacher's Word Book*. New York: Teachers College, Columbia University, 1921, 134 pp.

books examined were not the most recently published arithmetic textbooks and although the authors of the later books have given considerable attention to this matter of vocabulary, words which pupils do not know are still found too often in text materials which have been prepared for their use.

Hydle and Clapp found that success in problem solving depended in no small degree upon having an objective setting for the problem. Thus, a problem about rabbits was easier than a problem about hours of time, other things being equal.[9] They also found that problems having small numbers were easier than similar problems having large numbers.[10] These findings suggest that the teacher should proceed slowly from problems dealing with familiar, objective situations to those dealing with less familiar, subjective situations and that small numbers should be used long enough in problems of a given type for the pupils to develop competence in their solution before large numbers are used.

There is difference of opinion as to whether problems which are based upon familiar situations are easier for pupils than are problems which are based upon unfamiliar situations. Hydle and Clapp concluded that unfamiliar situations which are easily visualized are neither harder nor easier than familiar situations which are hard to visualize.[11] A carefully conducted experimental study of familiar *versus* unfamiliar situations in problems was made by Brownell.[12] This study failed to

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[9] Hydle, L. L. and Clapp, Frank L., *op. cit.*, p. 27.

[10] *Ibid.*, p. 34.

[11] *Ibid.*, pp. 62-63.

[12] Brownell, William A., with the assistance of Stretch, Lorena B.

reveal a significant advantage for the familiar situation. Commenting on the results of these two studies, White suggests that teacher opinion of familiarity was used in both and that such opinions may not have been valid. Experimenting with 1,000 pupils in grade 6B, she found experience with the situation involved in the problem to be important and especially important for the more difficult problems.[13] It is possible that pupils may find greater interest in problems which are based upon familiar situations, particularly those which are readily visualized because of their objectivity. If this is true, the less familiar situation and the less readily visualized situation should come later.

Lazerte made a number of scientific studies of problem solving.[14] One of his conclusions was that a rich background of experience in the situations with which problems deal is a very important aid to the pupils who undertake to solve them.[15] Furthermore, he found that boys had an advantage over girls, especially in the fifth grade, because the problems dealt more with boys' affairs than with girls' affairs.[16] There was no evidence that boys have more native arithmetic ability than girls have.

Teaching pupils to solve problems. Investigations of

The Effect of Unfamiliar Settings on Problem-Solving. Durham, North Carolina: Duke University Press, 1931, 86 pp.

[13] White, Helen M. "Does Experience in the Situation Involved Affect the Solving of a Problem?" Education: LIV: 451-455, April, 1934.

[14] Lazerte, M. E. *The Development of Problem Solving Ability in Arithmetic.* Toronto: Clarke, Irwin and Company, Limited, 1933, 136 pp.

[15] *Ibid.*, p. 124.

[16] *Ibid.*, p. 125.

what pupils actually do when they undertake to solve problems usually reveal results which are far from encouraging. Monroe made quite an elaborate study of the nature of pupils' responses in solving problems.[17] His subjects were 9,256 pupils in 41 Illinois cities. He recognized that solving a problem should be an exercise in reflective thinking.[18] However, most of the pupils seemed to "perform almost random calculations on the numbers given." This meant, of course, that many of their solutions were incorrect but when they were correct, they seemed to be more the result of habit than of reflective thinking. As long as the problems were stated in familiar terminology and there were no irrelevant data, they got along rather well but when unfamiliar terminology was used or new elements were introduced, they seemed to make no attempt to reason. Either they did not try to solve the problems or their solutions were incorrect.[19]

In making analyses of pupils' errors in problem solving, the author has found much evidence of what seemed to be purely random manipulation of the numbers which the problems contained. In many cases, there seemed to be no grasp of the conditions of the problem and no plan for its solution. In fact, in one study involving 1,196 errors made by 117 pupils in grades five, six, seven, and eight, 691 of the errors, or 57.8 per cent of the total, were classified as "procedure wholly wrong or entirely inadequate." These errors represented a

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[17] Monroe, Walter S. *How Pupils Solve Problems in Arithmetic.* Urbana, Illinois: University of Illinois, 1929, 32 pp.

[18] *Ibid.*, p. 7.

[19] *Ibid.*, p. 19.

great variety of wild guesses and random manipulations, such as finding the interest on $2,000 from January 15 to July 15 at 6 per cent by multiplying $2,000 by 15 and dividing the product by 6, or, finding the total number of yards of lace Mary sold, when she sold 2 yards to one customer and 1¾ yards to another, by multiplying 2 by 1¾.[20]

It was discovered that the teachers were responsible for some of these wild guesses for they had urged the pupils to "try" all of the problems including those which they could not be expected to solve. Thus, fifth-grade pupils whose work had included no reference to percentage or its application in interest "tried" the interest problem referred to in the preceding paragraph. This seems to be a common condition in some schoolrooms, particularly when tests or examinations are the order of the day. If, by chance, a pupil stumbles on to a correct solution, the teacher seems to share his satisfaction. If the solution of problems is to constitute practice in effective thinking, it would seem that the attainment of this important objective is rendered no more probable by such experiences.

It is obvious that many pupils solve problems by a trial-and-error procedure and that few of the trials are the result of reflective thinking. Stevenson suggested that pupils often decided what process to use from the form of the problem. One pupil said that he added because there were too many numbers for him to do anything else. He quoted a sixth-grade colored girl who

[20] Morton, R. L. "An Analysis of Errors in the Solution of Arithmetic Problems." *Educational Research Bulletin,* (Ohio State University) IV: 187-190, April 29, 1925.

described her method of solving problems as follows: "If there are lots of numbers I adds. If there are only two numbers with lots of parts (digits) I subtracts. But if there are just two numbers and one littler than the other, it is hard, I divides if they come out even, but if they don't I multiplies."[21] It is the author's observation that the use of this method is by no means confined to members of the colored race.

Bradford showed that if children are given a series of impossible problems, most of them undertake to write out solutions.[22] The five problems which he used included the following:

1. A boy is five years old and his father is 35 years old. If his uncle is 40 years, how old will his cousin be?
2. If Henry VIII had six wives, how many had Henry II?

Sixty-eight per cent of the twelve- and thirteen-year-old pupils worked out answers for the first of these problems and 58 per cent for the second. These were English pupils but it would be interesting to see what American pupils would do under similar circumstances.

Various investigators have undertaken to determine which is the best method for teaching problem solving. The methods are called by various names but they may be summarized conveniently as follows:

1. The conventional or formal analysis method.
2. The analogies method.
3. The individual method.
4. The graphic or diagrammatic method.

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[21] Stevenson, P. R. "Difficulties in Problem Solving." *Journal of Educational Research,* XI: 95-103, February, 1925.

[22] Bradford, E. J. G. "Suggestion, Reasoning, and Arithmetic." *Forum of Education,* III: 3-12, February, 1925.

The method of formal analysis. Where there has been any conscious attempt to develop a method in teaching problem solving, the method of formal analysis probably has been the most frequently used. Because of the frequency with which it is found, it is also called the "conventional" method. In its simplest form, this method consists of teaching the pupil to ask and answer these three questions if he is not sure that he knows how to solve the problem.[23]

1. What am I to find out?
2. What facts (numbers) are given?
3. What shall I do with the numbers?

Sometimes, these questions are elaborated and others are added. For example, Durell describes a method which includes six steps, as follows:[24]

1. State what is given.
2. State what is to be found.
3. Make a list of the operations to be performed.
4. Estimate the answer.
5. Make the computations.
6. Check the answer.

It will be seen that this outline of steps differs but little from that which is involved in the three questions proposed by Thorndike. Making an estimate of the answer and checking the answer are steps which may well be taken in connection with any method.

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[23] Thorndike, Edward Lee. *The New Methods in Arithmetic.* Chicago: Rand McNally and Company, 1921, p. 139.

[24] Durell, Fletcher. "Solving Problems in Arithmetic." *School Science and Mathematics,* XXVIII: 925-935, December, 1928.

The method of formal analysis seems to have an advantage in that it requires the pupil to read the problem critically and to think specifically of what is required and of what facts he is given. If these data are written down and properly labeled, it seems to be easier for the pupil to think of them with reference to the question which the problem asks. But the difficulty with the method seems to lie in the fact that there is a tremendous gap between seeing what is given and what is to be found and deciding what steps should be taken to solve the problem. In other words, the so-called formal analysis method can hardly be called a method at all.

A rather extensive investigation by Washburne and Osborne led to the conclusion that training in formal analysis had no appreciable effect upon the ability of pupils, especially the brighter pupils, to solve problems.[25] They got better results by simply having pupils solve many problems without the use of any particular method of attack.

The conventional method was one of three methods tested by Hanna.[26] He used fourth- and seventh-grade pupils of three levels of ability in arithmetic for a period of six weeks. The conventional method was found to be inferior to both of the other methods, although the differences were not large.

Two methods, one of which was called an analytical method and the other a non-analytical method, were

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[25] Washburne, Carleton W. and Osborne, Raymond. "Solving Arithmetic Problems." *Elementary School Journal,* XXVII: 219-226 and 293-304, November and December, 1926.

[26] Hanna, Paul R. *Arithmetic Problem Solving.* New York: Bureau of Publications, Teachers College, Columbia University, 1929, 68 pp.

used by Adams.[27] The non-analytical method was essentially the same as the method which we have been referring to as the conventional method. The analytical method went further; it provided for a greater degree of analysis, more detailed questions. The pupil was to determine what name his answer would have and whether the answer would be more or less than some number given in the problem. The results favored the analysis method slightly but there was little difference.

On the whole, the conventional method of teaching problem solving can not be recommended. There is little reason to believe that it will yield results better than those secured by mere practice and without the use of any specific method.

The method of analogies. The author's observations lead him to believe that the method of analogies is very commonly used by teachers in their efforts to get pupils to see the method of solution for a difficult problem. By this method, the pupil is given an easy oral problem which is similar to a difficult written problem. It is presumed that the pupil can solve the easy oral problem, that he will see the analogy to the difficult written problem, and then be able to solve the latter.

The analogies method was one of those used by Washburne and Osborne.[28] They found that pupils in the sixth and seventh grades made marked progress by the use of this method but that they made greater progress by the so-called individual method which will be de-

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[27] Adams, Roy Edgar. *A Study of the Comparative Worth of Two Methods of Improving Problem Solving Ability in Arithmetic.* Philadelphia: University of Pennsylvania, 1930, 68 pp.

[28] Washburne, Carleton W. and Osborne, Raymond, *op. cit.*

scribed later. They found the analogies method to be slightly superior to the method of formal analysis for the brighter pupils but decidedly inferior for those below the average in intelligence. It has already been indicated that the method of formal analysis was inferior to the individual method for all intelligence levels.

It hardly seems probable that the use of the analogies method will be very successful in stimulating the use of the higher thought processes. As Foran points out, the use of this method may result in the habit of associating problems with a certain type problem merely because there is a superficial resemblance. This does not stimulate thinking but may well discourage thinking.[29] The teacher's purpose in teaching problem solving should be the guidance of the pupil through the major difficulties which he encounters and the development in him of the ability to surmount such difficulties for himself. The method of analogies holds little hope for the accomplishment of such a purpose.

The individual method. The individual method is not a separate method. It is simply a name applied to the procedure of those who were left to their own devices. We have seen that Washburne and Osborne used the analogies method, the method of formal analysis, and the individual method.[30] The pupils described as using the individual method were those constituting the control group, that is, those against whose progress the progress of those using the method of formal analysis or

[29] Foran, T. G. "Methods of Teaching Problem Solving in Arithmetic." *The Catholic Educational Review,* XXXII: 269-282, May, 1934.

[30] Washburne, Carleton W. and Osborne, Raymond, *op. cit.*

the method of analogies was checked. In other words, the individual method is not a distinct method. The term is used by investigators to indicate the absence of a method.

Hanna also used three methods,—the conventional, the dependencies, and the individual.[31] Here again, the individual method was the method used by those in the control group and with whose progress the progress of those using the other two methods was compared. Hanna found the individual method to be superior to the conventional method; so did Washburne and Osborne, as has been stated.

The superiority of the individual method over the conventional method should not be taken as proof that the best way to teach problem solving is to leave the pupils to their own devices. Rather, it should be taken as evidence of the ineffectiveness of the conventional method. To leave pupils to their own devices is to deny that the teacher can do anything at all to help pupils learn to solve problems except to provide them with problems for practice.

The graphic method. Clark and Vincent experimented with a method which they called a graphical analysis method.[32] They compared results secured through the use of this method with those secured from the conventional method and found the former to be better. Pupils using the method of graphical analysis were more successful in determining what operations to

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[31] Hanna, Paul R., *op. cit.*

[32] Clark, John R. and Vincent, E. Leona. "A Comparison of Two Methods of Arithmetic Problem Analysis." *Mathematics Teacher*, XVIII: 226-238, April, 1925.

perform although they had no advantage in determining what the problem demanded. This method helped the duller pupils more than the brighter.

The graphical analysis method provides for analyzing the problem by means of a diagram. The writers use the following problem to illustrate the method.[88]

A grocer bought 24 bu. of potatoes at $1.50 per bushel. Four bushels spoiled. The others were sold at $2.00 per bushel. Find his profit.

- Profit
  - Cost
    - Number of bushels bought (24)
    - Price per bushel ($1.50)
  - Selling Price
    - Number of bushels sold
      - Number of bushels bought (24)
      - Number of bushels spoiled (4)
    - Price per bushel ($2.00)

The pupil is supposed to use the diagram to help him see that to find the profit, he must know the cost and the selling price; that to find the cost, he must know the number of bushels bought and the price paid per bushel; and that to find the selling price, he must know the number of bushels sold and the price received per bushel.

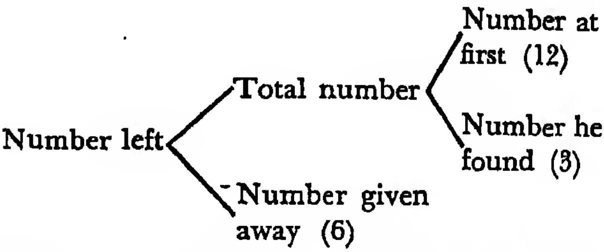
Hanna used a similar method which he called the *dependencies method*. After noting what is to be found, the pupil observes that the answer depends upon the data given in the problem, that these data may depend

[88] Clark, John R. and Vincent, E. Leona, *op. cit.*, pp. 226-227.

upon certain other data, etc. He illustrates with this problem.[34]

> Sam had 12 marbles. He found 3 more and then gave 6 to George. How many did Sam have left?

To find the number of marbles Sam has left, the pupil is supposed to think that he must find the total number of marbles Sam had and the number which he gave away. To find the total number of marbles Sam had, he must know the number he had at first and the number he found. Hanna suggests that these facts may be represented graphically as follows:



The pupil would start by writing "Number left" since this is what is to be found. Then he would recognize that this depends upon the total number and the number given away and would write these items in, connecting them with lines to "Number left" as shown. Since the number given away is known, 6 is written in parentheses. Then, recognizing that the total number depends upon the "Number at first" and the "Number he found," lines are drawn from "Total number" and these items are written in as shown. These facts are given, so 12 and 3 are written in parentheses. With the

[34] Hanna, Paul R., *op. cit.*, pp. 4-5.

analysis completed in this manner and the data recorded, the required computations are made.

Of course, the dependencies method can be used with or without the graphic representation although it would not be called a graphic method unless a diagram were used. In either event, the outstanding advantage of the method lies in the fact that it emphasizes the relationships between the given numbers. This emphasis seems to aid the pupil in arriving at a solution.

Hanna used the dependencies method with fourth-grade and seventh-grade pupils. The results were better than those secured from the use of the conventional method in each grade but not enough better to guarantee significance.

The value of visual aids in solving arithmetic problems is also emphasized by Otis.[36] He illustrates with the following problem.

If a stick 20 in. long is cut into two pieces so that one piece is $\frac{2}{3}$ as long as the other, how long will the longer piece be?

20 Inches
Scale 1 In.=8 In.

Since one piece is to be $\frac{2}{3}$ as long as the other, the longer piece may be represented by 3 segments and the shorter by 2. Then the entire stick whose length is 20 in. may be thought of as divided into 5 segments, each of which would be 4 in. in length. A drawing makes the remain-

[36] Otis, Arthur S. "The Visual Method of Solving Arithmetic Problems." *Mathematics Teacher*, XXI: 483-489, December, 1928.

der of the solution comparatively easy. Otis presents no experimental data on the value of such visual aids, but teachers who have used the plan testify as to its worth.

Which is the best method? It must be apparent to the reader that no one of the problem solving methods which have been tried under experimental conditions produces markedly superior results. However, since the periods of trial were brief, it is possible that there was insufficient time for the pupils to learn thoroughly the methods to which they, so far, were unaccustomed. It is possible that this has worked to the special disadvantage of the diagrammatic method since the conventional method has been widely used in textbooks in recent years and since teachers have not been at all unacquainted with the analogies method.[80] Nevertheless, the experiments which have been conducted give some slight advantage to the method of graphic analysis.

On theoretical grounds, the graphic or dependencies method holds much more promise than does any other method. It emphasizes significant relationships with a view to leading the pupil to see how these relationships can be used in arriving at a solution. It places a premium on thinking rather than on the mere perception of an analogy or the routine following of formalized steps. It gives the pupil something definite to do in developing an attack upon a difficult problem in contrast to the so-called individual method which leaves him entirely to his own devices.

But after all, there is no method of problem solving

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[80] Hanna reports that the conventional method is recommended in 14 of 20 textbooks that he examined and in 12 of 16 books on the teaching of arithmetic. Cf. Hanna, Paul R., *op. cit.*, pp. 50-51.

which can take the place of a good teacher. And there is no substitute for intelligence and hard work on the part of the pupil. As Monroe and Engelhart remark, after reviewing studies of problem solving methods, the effectiveness of any method depends to no small extent upon the skill and the zeal of the teacher who uses it.[37] The teacher whose enthusiasm stimulates the pupils to hard work and whose skill is sufficient to enable her to locate precisely where their difficulties lie and to assist them in surmounting these difficulties will do better with a poor method than will a poor teacher with a method which is known to be good. But the good teacher probably will do better if she will emphasize dependencies and represent relationships diagrammatically.

Special exercises to improve problem solving skill. Various schemes have been advanced for the purpose of helping pupils to overcome some phase of problem solving difficulty. Mitchell reconstructed five rather difficult problems by adding a series of analytical questions to be asked concerning each.[38] The results secured from 117 pupils in grades 7 and 8 were much better as a result of the use of this plan. Whether such an improvement is permanent and transfers to other problems, particularly those different in kind, is not known. However, it seems reasonable that pupils should learn to

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[37] Monroe, Walter S. and Engelhart, Max D. *A Critical Summary of Research Relating to the Teaching of Arithmetic.* Urbana, Illinois: University of Illinois, 1931, p. 67.

[38] Mitchell, Claude. "Problem Analysis and Problem Solving in Arithmetic." *Elementary School Journal,* XXXII: 464-466, February, 1932.

analyze problems to the extent that they see clearly what is required.

A program of special exercises was used in grades 4 to 8 for several weeks by Wilson.[39] She had pupils estimate answers and judge absurdities. The absurd answers used were those actually given by pupils who had taken a test. In another exercise, the pupils substituted other expressions for those which were underlined, such as *at the rate, total,* and *average* to determine whether their failures had been due to the presence of unfamiliar expressions. In another exercise, they read problems and indicated the processes which would be used to solve them. The result was a greater increase in scores on the Buckingham Scale for Problems in Arithmetic[40] than was made by a control group which did not have the advantage of the special exercises.

Specially constructed reading exercises are recommended by Brueckner.[41] He suggests three kinds of exercises to encourage the pupils to read problems carefully.[42] In one of these, a problem is given and is followed by several statements with instructions for the pupil to decide whether the statements are true or false. In another, an important fact is missing from the statement of each problem and the pupil decides what fact is missing. In the third, problems are given without

[39] Wilson, Estaline. "Improving the Ability to Solve Arithmetic Problems." *Elementary School Journal,* XXII: 380-386, January, 1922.

[40] Buckingham, B. R. *Scale for Problems in Arithmetic.* Bloomington, Illinois: Public School Publishing Company.

[41] Brueckner, Leo J. "Improving Work in Problem Solving." *Elementary English Review,* VI: 136-140, May, 1929.

[42] Brueckner, Leo J. *Diagnostic and Remedial Teaching in Arithmetic.* Philadelphia: The John C. Winston Company, 1930, pp. 315-317.

numbers and the pupil states the process which would be used in solving each problem. An example of the last is, "If you know the price of sugar a pound, how would you find the number of pounds you can buy for a dollar?" Brueckner states that the use of reading exercises results in marked growth in the ability to solve problems.[43]

Many writers recommend that pupils should estimate the answers to problems before solving them. The purpose, apparently, is to lead them to reject absurdly impossible answers if they are obtained. The author recalls watching a pupil obtain the answer, $67.50 as the price of a pair of shoes. He checked through his computations and, apparently, was completely satisfied with his answer. When his teacher inspected his work, she made no effort to make him skeptical as to the answer, $67.50, for the price of a pair of shoes. She simply said; "Your work is all right, John, except that you have the decimal point in the wrong place." Naturally, her correction had very little permanent effect.

A limited experiment on the value of estimating answers has been reported by Dickey.[44] According to his results, the sixth-grade group which estimated answers apparently did no better than another group which followed the traditional practice. Of course, this does not mean that answers which are absurdly at variance with the conditions of the problem or with ordinary experience should be tolerated by either the teacher or the

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[43] *Ibid.*, p. 328.

[44] Dickey, John W. "The Value of Estimating Answers to Arithmetic Problems and Examples." Elementary School Journal, XXXV: 24-31, September, 1934.

pupils. It is often quite difficult for a pupil to make a reasonable answer for a problem in advance of his efforts to solve it.

Much emphasis is placed upon cues by Osburn.[45] Just as an actor in a theater knows what he is to do or say when his cue comes, a pupil would take his cue as to the operation to be performed from the wording of the problem.[46] The total list of cues found in a sampling of textbooks is very long. For multiplication alone, 190 cues were found. Osburn suggests that we should select from these cues those most frequently used and teach them well. For multiplication, he gives an abbreviated list of 59 cues for use in grades 3 to 6 and selects 30 of these which he would have taught more carefully than the others.[47] The first three of these in the third grade list are stated as follows:

1. How much do ____ articles cost at ____ cents each?
2. If I can pick ____ qt. (bu., etc.) of berries (etc.) in one day, how many quarts (bushels, etc.) can I pick in ____ days?
3. What is the cost of ____ dozen oranges at ____ cents a dozen?[48]

There is no doubt that pupils do make use of cues and that teachers encourage their use. One pupil whom the author asked how he knew he should multiply in solving a problem, said, "Whenever it says 'at,' I mul-

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[45] Osburn, Worth J., *op. cit.*, pp. 12-25 and 205-234.
[46] *Ibid.*, p. 12.
[47] *Ibid.*, pp. 224-226.
[48] *Ibid.*, p. 224.

tiply." Apparently, this pupil had not thought about the problem at all. His reactions were as mechanical as if he had been solving a series of multiplication *examples.* This suggests the weakness of the cue approach to problem solving. Kramer suggests that the cue approach does not stimulate thinking, that problems are solved mechanically rather than intelligently if the solutions are simply learned reactions to familiar cues.[49] We have indicated frequently that the meaning theory should be the foundation of arithmetic instruction. In problem solving, particularly, the meaning theory should predominate.

In some schools it is the custom to require that all solutions be written out in accordance with a set form. If two or more solutions are possible, none is accepted except that which has been prescribed. For example, a class was given the following problem as one of several which were similar.

> Dick's mother has a rug 9 ft. by 12 ft. in the living room. The room is 18 ft. long and 14 ft. wide. Next Saturday, Dick is going to varnish the strip of floor around the room which the rug does not cover. The man at the paint store told Dick that a pint can of varnish would cover 150 sq. ft. Will a pint be enough?

The teacher had solved a similar problem on the blackboard, making a drawing and showing that there were four strips to be varnished. Two of these strips ran the full length of the room and two others ran the width of the room less twice the width of the lengthwise strips.

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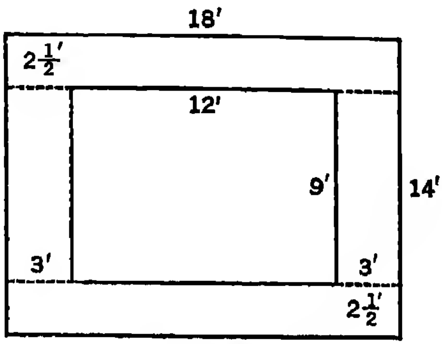
[49] Kramer, Grace A. "Some Wrong Notions in Devising Arithmetic Problems." Education, LIV: 473-476, April, 1934.

In solving the above problem the pupils were expected to make a drawing and find the areas of the strips, as shown. Each of the two long strips would then have an area of 45 sq. ft. and each of the short strips would have an area of 27 sq. ft. Then the entire area to be varnished would be 144 sq. ft.



Scale 1 in.=9 ft.

Obviously, there is another solution. One pupil—a very capable one—saw this solution and presented it the next day in class. He said: The area of the whole floor would be 252 sq. ft. The area of the part covered by the rug would be 108 sq. ft. Then, you would have to varnish 144 sq. ft. So a pint can would be enough." The teacher rejected his solution and insisted that the other be given.

A pupil should be given credit for any solution if it is sound and if he gets the correct answer. If his solution is long, laborious, and time-consuming, he should be shown that there is a shorter and easier way. But to discount the work of a pupil who thinks out a different solution for himself because his solution does not con-

form to the model which we have set is a short-sighted policy indeed when we are trying in problem solving to teach pupils to think for themselves. In actual life problems it is the answer which counts, together with assurance that the answer is correct, regardless of how it was obtained. Thorndike says, "Concerning the technique of solving problems and expressing the solution, the newer methods advocate extreme catholicity."[50]

Summary of suggestions for teaching problem solving. Any serious effort to outline a program for the teaching of problem solving will leave the reader with a feeling of disappointment. We have seen that this is a phase of arithmetic teaching concerning which not very much is known. There is scant experimental evidence to support any recommendation. What evidence there is fails to agree in several important respects. In this chapter, an effort has been made to review the best of the literature on the subject and to offer suggestions, partly colored by the author's own experience and point of view, for the teacher's guidance. The teacher will find, nevertheless, that much depends upon her own insight and skill and upon her ability to stimulate the pupil to make the best possible use of the intelligence which he possesses.

The main recommendations may be summarized briefly as follows.

1. Standardized tests should be used at the beginning of a program of instruction and occasionally thereafter to determine the attainments of the pupils and their capacity to learn.

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[50] Thorndike, Edward Lee, *op. cit.*, p. 138.

2. The chief factor conditioning success in problem solving is intelligence. Therefore, assignments should be varied to suit the varying capacities of the members of a class. The teacher should recognize that dull pupils who are unable to solve the problems in the assignment for the grade in which they are classified may be able to solve those problems when they have become intellectually more mature.

3. Success in solving problems is rather highly correlated with skill in the basic operations. Therefore, pupils must become proficient in these operations.

4. Problems solved in school come largely from printed sources. Hence, pupils must be skilled in reading.

5. The terms and vocabulary used in statements of problems should be those with which the pupils are acquainted. Necessary new terms should be taught before they are used and should be practised until they are known.

6. Problems dealing with relatively unfamiliar or subjective situations should be used later than those dealing with more familiar or objective situations.

7. Problems in which large numbers occur should be used later than similar problems which use small numbers.

8. Teachers should not urge pupils to try to write out solutions for problems which they can not solve. Pupils should be encouraged and stimulated to make their best efforts but bluffing is not a substitute for reflective thinking.

9. The conventional or formal analysis method may

encourage pupils to read problems carefully but there is little evidence that it is an effective method for arriving at solutions.

10. The method of analogies increases the pupil's chances of success as long as the problems conform to patterns which have been learned but this method does not seem to be an effective means for stimulating thinking.

11. The so-called individual method is not a method at all. Progress credited to the use of this method probably is merely the progress which is accomplished through unguided practice.

12. The graphic or diagrammatic method seems to produce results better than those secured through the use of any other method which has been tried under experimental conditions. Its advantage seems to lie in the fact that the visual aid which it supplies increases the probability that the pupil will analyze the problem successfully and perceive the necessary relationships.

13. Special exercises designed to help the pupil read a problem critically and understandingly, to perceive absurdities, and to look for necessary data, seem to help him in arriving at a solution.

14. Special emphasis upon cues, like the use of the method of analogies, does not seem to encourage the use of the higher thought processes.

15. Next to the possession of intelligence by the pupil, probably the most important single factor in developing the ability to solve problems is skill and insight on the part of the teacher. No method can substitute for a good teacher.

QUESTIONS AND REVIEW EXERCISES

1. Which is more important, the ability to solve problems or the ability to make computations? Why?

2. If the ability to perform the fundamental operations is a means to an end and the end is problem solving, can problems be used to introduce new types of examples?

3. What distinction is made in this chapter between a problem and an example? Can a problem be an example? Can an example be a problem?

4. What is the chief factor conditioning success in problem solving? What other factors rank high in importance?

5. What is the correlation coefficient which expresses the relation between two variables if these variables are perfectly related in a positive sense? If they are perfectly related in a negative sense? If they are not related at all?

6. Is reading an important factor in solving problems in school? Why? In solving problems outside of school? Why?

7. Do you believe that there is a positive correlation between children's scores on a test in problem solving and the incomes of the fathers of these children? Why?

8. How do you account for the very low correlation between ability in problem solving and school attendance?

9. Can dull children learn to solve difficult arithmetic problems? What should be the teacher's attitude toward dull children as regards problem solving?

10. Some teachers object to the criticisms which have been made of the use of difficult or unusual words in arithmetic problems on the ground that one objective of the arithmetic teacher should be the enlargement of the pupil's vocabulary. Discuss.

11. What should be the effect upon the course of study of a finding to the effect that problems having a familiar objective setting are easier than those having a less familiar

and subjective setting? Give examples of problems in each of these two classes.

12. How do you account for the seemingly random calculations of pupils who are trying to solve problems?

13. Which would be better as a part of an intelligence test, a group of arithmetic problems or a group of arithmetic examples? Why?

14. What is the conventional or formal analysis method of problem solving? What merit does the method have? What are its weaknesses?

15. What is the method of analogies? Point out the good qualities and the poor qualities of this method.

16. What is the graphic or diagrammatic method? Can it be used equally well for all problems? How does it resemble what Hanna calls the dependencies method?

17. Why is it stated in this chapter that the individual method is really not a method at all?

18. In the light of the discussion in this chapter, which is the best problem solving method? Has evidence been assembled to the effect that this method is undoubtedly better than any other method?

19. Which is the more important, a good method or a good teacher? Why?

20. What benefit is there in your *opinion* in including unnecessary or irrelevant numbers in a problem?

21. Describe exercises which you would construct for the purpose of teaching pupils to read problems more carefully.

22. What were Dickey's findings as to the advantage of estimating the answer before solving the problem? How can pupils be expected to recognize absurd answers if they obtain them?

23. What use would you make of cues in teaching problem solving? Why?

24. As a summary of the chapter, make a list of the

suggestions which you would give a group of teachers to help them in teaching problem solving.

CHAPTER TEST

For each of these statements, select the best answer. A scoring key will be found on page 534.

1. Pupils' first experiences in problem solving should come (1) before they have acquired the needed basic skills (2) while they are acquiring the needed basic skills (3) after they have acquired the needed basic skills.

2. Find the square root of 500. This is (1) a problem (2) an example (3) neither a problem nor an example.

3. Problems are more difficult than examples (1) always (2) sometimes (3) never.

4. The factor "skill in the fundamental operations" correlates rather highly with "ability in problem solving" because (1) it is used in problem solving (2) it is taught first (3) it increases the ability to solve problems.

5. Ability in problem solving was found to correlate most highly with (1) verbal intelligence (2) reading comprehension (3) skill in the fundamental operations.

6. The largest possible positive correlation coefficient is (1) .90 (2) 100 (3) 1.00.

7. The correlation between school attendance and problem solving ability was found to be (1) very low (2) moderately high (3) very high.

8. Of the three factors, intelligence, skill in the fundamental operations, and reading comprehension, the least modifiable is (1) intelligence (2) skill in the fundamental operations (3) reading comprehension.

9. In the solution of life problems, reading comprehension is (1) less important (2) equally important (3) more important than in the solution of typical school problems.

10. A child with an I. Q. of 85 can be expected to solve typical fifth-grade problems (1) when ten years of age (2) when twelve years of age (3) not at all.

11. In teaching problem solving, the enlargement of a child's vocabulary should (1) be a major objective (2) be a minor objective (3) not be considered at all.

12. *The Teacher's Word Book* was prepared by (1) Brueckner (2) Brownell (3) Thorndike.

13. The problem solving method found most frequently in textbooks and books on the teaching of arithmetic by Hanna was (1) the conventional method (2) the graphic method (3) the method of analogies.

14. Those who have investigated the efficacy of the conventional method (1) disagree markedly as to its value (2) tend to agree that the method is effective (3) tend to agree that the method is not very good.

15. On the whole, the best method seems to be (1) the conventional method (2) the analogies method (3) the graphic method.

16. The graphic method was investigated by (1) Hydle and Clapp (2) Washburne and Osburne (3) Clark and Vincent.

17. In the discussion of this chapter, the dependencies method was classified with (1) the conventional method (2) the analogies method (3) the graphic method.

18. Compared with methods of problem solving, good teaching is (1) more important (2) equally important (3) less important.

19. The teaching of a list of cues (1) was recommended as a means of stimulating thinking (2) was recommended for dull pupils only (3) was not recommended.

20. The use of standardized tests in connection with the teaching of problem solving (1) was recommended (2) was not recommended (3) was not mentioned.

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CHAPTER 12

THE USE OF TESTS AND EXAMINATIONS

In recent years, much has been written and much more has been said about tests and examinations in arithmetic and other school subjects. Prior to the early years of the twentieth century, there was little disposition to question the value or significance of examinations. The results obtained from the use of examinations were taken at their face value for the simple reason that no one had gone to the trouble to determine whether or not those results were valid. Even today there are many teachers in elementary and secondary schools and in colleges who prepare examinations on short notice and who score the resulting papers with supreme confidence in the accuracy of their judgments.

Common defects in examinations. That examinations as prepared and used by many teachers are seriously defective is easily shown. In the first place, if two or more teachers grade the same examination paper, and do not confer on the subject, they frequently disagree markedly as to what the paper is worth. Thus, the mark which a pupil receives on an examination depends not only on what he writes on his paper but also on the individual who grades the paper. Although the objective nature of arithmetic examples and problems has led many teachers to believe that grades in this subject are likely to be reliable, variation from teacher to teacher in marking the results of arithmetic examinations has been found to be almost as great as variation in marking the results of examinations in other school subjects.

Furthermore, papers are not rated consistently by the same teacher. It has been found that if a teacher scores a set of papers, makes a record of the marks on a separate sheet of paper, and rescores them a few weeks later when the first set of marks has been forgotten, there are frequently large differences between the two marks given the same paper.

In the second place, it is contended that no single examination can measure adequately a pupil's grasp of a subject. Serious injustice may be done by basing promotion upon the results of a single examination. This is particularly true if the examination is one of the essay type or, in arithmetic, one consisting simply of a series of problems and examples to be solved. Because of limitations of time and energy, only a sample of all that has been taught may be included in the examination. Final examinations at the end of a semester's or a year's study may include only a small sample of all that has been taught. Although this sample may give a fairly satisfactory measure for the class as a whole, it may give a very unsatisfactory measure for individual members of a class, underrating some and overrating others.

In the third place, it is said sometimes that examinations are out of harmony with desirable educational objectives. Teachers complain that they can not eliminate the unreal problem, the formal definition or the topic of doubtful social value for fear that these things will be called for in examinations prepared by supervisors or others. Hence, teachers and pupils work to get ready for examinations and they judge the value of what has been taught and learned by the marks earned

on these examinations rather than by their usefulness in social situations.

There are many other objections to traditional examinations. It is claimed that although they may reveal the fact that a pupil is weak, they fail to show where his weaknesses are; that they are not diagnostic, in other words. It is said also that examinations neglect the rate at which pupils work although rate may be an important dimension of a pupil's ability; that questions of greatly varying value and difficulty are given equal weight in scoring; that examinations encourage the cramming of details rather than the understanding of broad principles; that examinations put a premium upon dishonesty in the form of cheating; that they are injurious to the pupil's health; that grading papers requires too much of the teacher's time; etc.

So vigorous has been the criticism of examinations that many teachers and others have come to doubt whether they should be used at all. It is sometimes suggested that, in the long run, better work can be done without examinations than with them. However, there are many who believe that there are so many benefits to be derived from the use of examinations that they should not be discontinued but should be improved so as to rid them of the objections which have been raised. Such persons do not contend that the objections are unsound but they assert that the defects which give rise to these objections can be removed or considerably reduced without detracting from the benefits which examinations yield.

The benefits of examinations. There are a number of advantages which may come from the use of examina-

tions if examinations are well prepared and are properly administered. The preparation of examinations will be discussed on later pages. The chief advantages may be stated as follows.

In the first place, the examination furnishes valuable information. Through its use, both the teacher and the pupil can be informed as to what the pupil knows and can do. Information furnished by an examination, particularly if the examination is diagnostic in character, may result in significant changes in later teaching plans. A good examination not only tells whether a pupil is weak or strong in a subject but also it tells where his weaknesses are. Thus, a good examination in the addition of common fractions will be constructed so as to reveal to the teacher whether or not the pupil has mastered the various elements of difficulty in this subject and precisely which elements, if any, he has failed to master.[1]

A good examination may reveal both a pupil's *relative* ability and his *absolute* ability. By *relative* ability is meant the ability of the pupil with reference to the abilities of other members of the class or a larger group —whether he ranks first, tenth, eighteenth, etc., or whether he is above or below the average. By *absolute* ability is meant the amount of ability which the pupil possesses above zero ability, or no ability at all. Examinations prepared by teachers and supervisors ordinarily do not reveal the pupil's absolute ability. Ordinarily, this is revealed through the use of specially prepared standardized scales.

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[1] See Chapter 5 for a discussion of the addition of common fractions.

Secondly, through the information which they supply, or it is thought they may supply, examinations give both teachers and pupils a *motive* for better work. It is human nature to want to do well on a test, to stand well with reference to others or to one's own previous record. Teachers, naturally, want their pupils to make a good showing and it is quite right that they should if the examination is a good examination. Thus, the examination is a stimulus both to teachers and to pupils.

A third benefit of the examination lies in the fact that it constitutes a good review lesson. Material which might otherwise be forgotten is thus met another time. Furthermore, pupils are stimulated to review in preparation for the examination and, with the teacher's help, they may organize better the content of the courses which they study. Monroe summarizes the benefits of such reviews as follows: "Even the intensive cramming which is deplored by many persons has educational value. If a student studies throughout the course, an intensive review and organization of the material is extremely valuable. In fact, the profit to the student is probably relatively greater for time expended in this work than for any other equal amount of time devoted to the subject. Unless carried to an extreme, cramming is undesirable only when it has not been preceded by thoughtful study. Even in this case it is better to have the student 'cram' for the examination than to go through the course without engaging in any learning. Incidentally, it should be noted that the existence of 'cramming' is not the fault of the examinations but of the type of instruction given and of the policy of making the student's final grade depend wholly or very largely upon

his examination mark. . . . The writing of an examination may itself be an important part of the student's learning. 'There is no impression without expression.' The writing of a three hour examination is undoubtedly an intensive form of expression. In case the questions are such that the reasoning and organization of information are required, the student may learn a great deal from the act of writing his answers to the questions. From this point of view, it may be urged that a student who is not required to take final examinations is deprived of an important opportunity for learning."[2]

Monroe was writing with particular reference to the needs and experiences of students older than those in the intermediate grades but the teacher in the intermediate grades also may secure real benefits from the use of examinations. To secure them, however, the teacher must prepare for an examination lesson with painstaking care. The preparation of an examination lesson probably should require more time and energy of the teacher than the preparation of any other lesson in the course. On the contrary, some teachers who give examinations to their pupils make no advance preparation whatsoever for the occasion. The author has seen teachers turn the pages of the textbook searching for problems to be solved, items of information to be called for in questions, and terms to be defined, after the hour for the examination had arrived and the pupils with paper on their desks and sharpened pencils in hand were ready for work. One by one, the questions were written on the blackboard, the teacher whirling occa-

[2] Monroe, Walter S. *Written Examinations and Their Improvement.* Urbana, Illinois: University of Illinois, 1922, pp. 24-25.

sionally to catch any would-be miscreant in an effort to give or receive help. All of the ills of examinations which have been enumerated—and more—are likely to be present in such cases.

Qualities of a good examination. We turn now to the important question, "How are good examinations constructed?" or, "What are the qualities of a good examination?" Some of the same criteria which are commonly used in determining the value of a "standardized" test may be applied to examinations which the teachers themselves prepare. They are:

1. Objectivity. Is the examination so constructed that the personal judgment or bias of the scorer is not an important factor in determining the mark which the pupil receives? To what extent would two or more persons secure the same marks if they were to score the papers of a class?
2. Validity. Does the examination really measure what it purports to measure?
3. Reliability. Do two or more forms of the examination or two halves of the examination yield consistent results?
4. Ease of giving and scoring. Must the examination be given by a person especially trained for this purpose? Is it difficult to score the papers?

We shall discuss briefly each of these four qualities.

Objectivity. It has been suggested that the mark which a pupil receives on an old-style examination depends upon two factors—first, what he writes on his paper and, second, the person who grades it. Manifestly, the second

factor should not count, but, in some cases, it determines a pupil's mark as much as does the first factor. Broad, general questions beginning with "Discuss," "Describe," "Tell about," etc., usually call forth equally broad, general answers. Obviously, an examination calling for such responses is not objective.

Non-objective tests encourage bluffing by verbose individuals who are deficient in knowledge of the subject. If the schools exist for the purpose of training children for effective participation in the affairs of a desirable social order, their purpose is defeated if tests which encourage this questionable type of ethics are used. An interesting illustration of bluffing came to the writer's attention a few years ago. On a teacher's examination in physiology and hygiene, the following item occurred: "Discuss: 'Expectoration in public places is to be condemned as a disgusting habit.' " A young aspirant for a teacher's certificate was stopped by this question because he did not know the word, *expectoration*. He knew, though, that to leave the question unanswered meant a deduction of 10 per cent from his score and he had reason to think that his paper could not stand this 10 per cent loss. *He wrote two pages in answer to this question.* He generalized and moralized on bad habits in a very broad and comprehensive manner; he then cited the habit of *expectoration* as a particularly unfortunate habit for one to possess; and he finally reached the conclusion that in public places especially the habit of expectoration was highly disgusting and should be condemned most heartily. Since the answer was long and was written in a fluent and

persuasive style, the member of the board of examiners who scored the paper allowed the item 10 per cent, a perfect score.

There are two rules which should be followed by those who must take old-style examinations. They are:

1. Answer every question, whether you know anything about it or not.

2. The less you know about a question the more you should write.

That is, these rules should be followed if one would make a good mark on the examination. In most old-style examinations, it pays to bluff. Bluffing never detracts from one's score in such an examination; it may increase it quite materially.

When examinations of the essay type are employed, those who rate the papers must use their judgment in determining the worth of an answer. But people's judgments do not agree; hence, essay type examinations ordinarily are not objective. In another county teacher's examination the following question and answer occurred in arithmetic.

Question: Explain as to a class the rule for ascertaining the area of a trapezoid or the volume of a cone.

Answer: The sides which are parallel to each other —Make a drawing illustrating the sides and altitude. (Trapezoid) Add the sides together and divide by 2—this is to get the average length of the sides. Show them that it is finished by (multiplying this by the altitude) and this is now the same as finding the area of a rectangle.

The English of this answer is poor as to punctuation and sentence structure but it is evident that the examinee knows something about trapezoids and could probably find the area of one. It is not evident that he would be able to make a good class presentation of this subject. What credit should this answer receive, if a perfect score is 10 per cent?

Fifty-five county school examiners, representing the boards of examiners of 55 different counties in one state, graded this answer. Each had been supplied with a mimeographed copy of the question and answer as given above. One of the 55 examiners gave this answer a score of 0; nine others scored it, 10; the remaining 45 gave marks between these extremes.

To be perfectly objective, a question must permit only one correct answer and the correctness of this answer must be obvious to one who knows the subject. Many of the new-type examinations are perfectly objective or nearly so. They will be explained and illustrated in later pages.

The essay type of examination is usually anything but objective. But the subjectivity of essay examinations need not be as great as it is. Ruch suggests that the variability in scoring such examinations may be reduced fifty per cent by agreeing upon and following scoring rules.[3] Kelly had six fifth-grade teachers give the same examination in arithmetic to their pupils and then had them score the papers. Later they rescored the papers using a set of scoring rules which had been prepared for them. The result was a marked decrease in the vari-

[3] Ruch, G. M. *The Objective or New-Type Examination.* Chicago: Scott, Foresman and Company, 1929, p. 105.

ability of the scores.[4] A similar experiment was carried out by O. W. Fauber. He had an arithmetic paper graded by 40 teachers without any specific rules for scoring and then by 48 teachers with a set of scoring rules. The result was a reduction of about one-half in the variability of the scores.[5]

Validity. If an examination is valid, it measures what it purports to measure. A valid examination is one which actually tells whether the pupil has the knowledge, skills, and attitudes which the course aims to develop. It must be evident, then, that the items which appear in the examination must be important items of the course. The inclusion of irrelevant matter or the failure to touch upon the outstanding facts, skills, and principles which have made up the course will tend to reduce the validity of the examination.

The validity of an examination is best determined by statistical methods, particularly by correlating the resulting scores with some acceptable criterion or standard of judgment, or by finding the per cents of pupils who were able to answer each of the items correctly at the various age levels. Such methods require more time than the teacher usually can give to the preparation of examinations. They also require a knowledge of statistical techniques with which the teacher may not be acquainted. Ordinarily, one must estimate the validity of an examination by checking it against the opinions

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[4] Kelly, Frederick James. *Teachers Marks: Their Variability and Standardization*. New York, Teachers College, Columbia University, 1914, pp. 83-84.

[5] Reported by Ruch, G. M., *op. cit.*, pp. 102-105.

of experts, especially those who have had a distinct influence upon educational practice through their published books and articles, courses of study, committee reports, etc.

In many cases, teachers can greatly improve the validity of their examinations by giving more thought to their preparation and by studying the published works of experts on aims and accomplishments in the subject. For example, a sixth-grade teacher prepared an examination in multiplication and division with decimals. She selected with care the numbers which should be included in the examples so as to provide for certain decimal variations in multiplicand, multiplier, product, dividend, divisor, and quotient. When she had finished, she recalled that some of the standardized tests in decimals which she had used gave the digits of the products and quotients and required the pupil merely to insert decimal points and write zeros where necessary. A little reflection convinced her that the test which she had prepared was not so much a test in decimals as a test in multiplication and division. She then greatly improved the validity of her examination by working out the products and quotients and giving the pupils a limited amount of time to write in needed zeros and to insert decimal points.

To be valid, an examination must cover thoroughly the significant details of the subject. If we would measure a pupil's ability in multiplication with decimals, for instance, we should include in the test all of the essential varieties of examples. Because a pupil can solve

the example, $\frac{428.15}{2.62}$, for instance, we can not be sure that he can solve $\frac{7.5}{.006}$, or $\frac{.025}{40}$.

A valid examination not only includes the important items of subject matter but it includes them under circumstances analogous to those under which they will normally be used. It is a much better test of a pupil's knowledge of plane figures, for example, to let him indicate which of a group of figures are triangles, which are rectangles, which are circles, etc., than to ask him to *define* triangle, rectangle, circle, etc. The pupil who has memorized definitions of *proper* and *improper* fractions but who cannot tell which of a number of fractions are proper and which are improper knows far less of the subject than the pupil who can readily identify these classes but who is unable to give acceptable definitions. Rarely, indeed, is an examination which calls for memorized definitions a highly valid examination.

Reliability. If the same pupils are tested twice by the same examiner with the same test, the reliability of the test is the extent to which the results of the two testings agree. The reliability is determined by correlating the two sets of scores. The resulting correlation coefficient is called the "reliability coefficient."

Reliability is often measured by correlating the scores obtained on two different tests rather than those obtained by giving the same test twice. If we prepare two examinations and try to make them equally good measures of the abilities in question, the reliability of either examination may be judged by measuring the extent to which the results secured through its use agree with the

results secured through the use of the other, that is, by finding the amount of correlation between the two sets of scores.

Many teachers do not recognize the distinction between reliability and validity. It should be noted that if a test gives consistent results it is reliable no matter whether or not it measures what it is supposed to measure; but a test is not valid unless it measures what it is supposed to measure. An examination may be reliable but not valid. But it can not be valid without being reliable.

To increase the reliability of an examination, we may first provide that it be objective. Objectivity has been discussed in a preceding section.

Another method of increasing the reliability of an examination is to increase its length. Other things being equal, the longer a test, the greater its reliability. The results on two addition tests, made up of examples of the same kind and of equal difficulty, will agree more closely if each test contain ten examples than if each test contain only five examples. A short test will be much more reliable for a class than for an individual in the class. That is, if one wished to compare two fifth-grade classes in their ability to add fractions, a short test would yield a reliable comparison but if one wished to compare two pupils in a class, a much longer test would be necessary. This is because the individual scores are combined to make a class score and chance errors tend to balance each other. The effect is the same as the effect of increasing the length of the test for an individual.

To increase the reliability of the marks which we give

pupils we may either increase the length of the test or give several tests and combine the results. A combination of these two methods probably should be used.

An examination can often be made more reliable by using fine rather than coarse units of measurement. If an examination contain five items or questions, weighted equally, an error or misunderstanding in any one of them may affect the score to the extent of 20 per cent. But if the same material be incorporated in ten items, the same error or misunderstanding may not affect the score by more than 10 per cent. The necessity of giving partial credits on answers involves personal judgment and thereby decreases both the objectivity and the reliability of the examination.

Consider the following problem as a part of an examination.

Two girls made three gallons of lemonade which they sold at five cents a glass, two glasses to a pint. They used 20 lemons at 45 cents a dozen, three pounds of sugar at seven cents a pound and 10 cents worth of ice. How much profit did each girl receive?

The problem calls for one bit of information for an answer, namely, each girl's share of the gain. A single computational error, or the lack of a bit of information, such as the number of pints in a gallon, will be as serious, so far as the pupil's mark is concerned, as the utter inability to think out the steps in the solution of the problem, unless we undertake to give partial credits. We can obtain a more reliable measure of the abilities of the individual members of the class if we ask *several* questions, such as, What did the lemons cost? How many

glasses of lemonade were sold? How much money was received? etc.

We can further increase the reliability of a test if we can remove or avoid certain variable conditions found among the pupils. It is best not to examine a pupil if he is suffering from a severe toothache, if the sun is shining directly on the paper on his desk, if he has been out quite late the night before, if his parents have quarreled at the breakfast table, etc. Some of these variable factors are beyond the knowledge or control of the teacher; others may be regulated.

Ease of giving and scoring. Much more labor is required for the preparation of the new-type examinations described in later pages than for the preparation of essay examinations, but time and energy are saved in scoring papers. The author has scored 40 "true-false" test papers, each containing 50 items, converted the scores into per cents, alphabetized the papers, and copied the marks in a class record book in one hour. The labor, after the scoring-key has been prepared, is quite routine in character and can be performed by a clerk.

The better examinations are easily given and easily scored. The hours which teachers spend in reading the tedious papers of pupils who have taken essay examinations often could be used in more profitable ways. Such labor consumes the teacher's energy, uses the time which might be spent in making constructive plans, and may do the pupil little good because the resulting scores may be unreliable and, therefore, invalid.

New-type examinations. There are four common kinds of new-type tests or examinations.

1. Alternate response tests.
2. Multiple response tests.
3. Matching tests.
4. Completion exercises.

Some of these occur in varied forms and are sometimes given special names. The chapter tests in this book include three of these four kinds of tests.

Alternate response tests. The commonest of the alternate response tests is the true-false test. This type of test consists of a series of statements, some (usually about one-half) true and some false. A true-false test may be mimeographed with the words, "True" and "False," after each statement. The pupil then underlines the word "True" when he believes an item to be true and the word "False" when he believes an item to be false. A common practice, also, is to omit the words, "True" and "False," and to instruct the pupils to place plus signs (+) before true statements and minus signs (—) or 0's before false statements. The words, "Yes" and "No," are sometimes used instead of "True" and "False."

In constructing true-false and similar tests, the teacher should go over the work carefully and prepare a list of statements covering the work in considerable detail. There should be no systematic order in arranging the true and the false statements but there should be approximately the same number of each when the examination is completed. The truth or falsity of the statements should not be obvious to one unacquainted with the subject if they are to be valid for testing purposes. Such statements as, "The divisor is one of the

numbers used in dividing," or, "There are just 50 feet in a mile," are of little or no value.

In writing the statements, great care should be exercised to see that the language used is language which the pupils can understand and that the statements are not ambiguous. One of the commonest faults of true-false tests, especially those which are the work of beginners, is this matter of ambiguity. It is sometimes found that the pupils read into the statements meanings quite different from those which the teacher had in mind. After a list of statements has been prepared it is well to have it criticized by other teachers before it is used.

If it is not possible to have the statements duplicated so that each pupil may have a copy, they can be written on the blackboard, or they can be written by the pupils from dictation, or the pupils can simply write the words, "Yes" or "No," in numbered blanks as the statements are read. Writing the statements is time-consuming and the latter plan is open to the objection that the pupils do not have sufficient opportunity to reflect on the more difficult items. For the most satisfactory results, the statements should be duplicated.

One of the duplicated copies may be marked by the teacher and used as a scoring key. The scoring of the papers is very rapid and easy for the scorer does not have to read the statements but simply compares the pupils' papers with the key. Since there are only two possible responses for each statement, a pupil can be expected in the long run to get about one-half of his guesses right and one-half of them wrong. The score, then, is usually the number right minus the number

wrong, although some contend that a more reliable measure is obtained by using the number right as the score. Of course, the right-minus-wrong method yields lower scores and the results must be interpreted accordingly.

The following true-false statements will illustrate this type of examination. The statements pertain to the mensuration of simple plane figures.

1. The sides of a square are equal..........True False
2. All rectangles have square corners......True False
3. All rectangles are squares..............True False
4. All squares are rectangles..............True False
5. To find the area of a triangle, we multiply the base by the altitude..........True False
6. A triangle has four sides..............True False
7. The area of a rectangle is found by using the formula, $A = lw$...........True False
8. The diagonal of a square divides the square into two triangles............True False

These eight statements do not make a comprehensive examination of the elements of mensuration; they merely illustrate how such examinations may be prepared.

Multiple response tests. In the multiple response test, as the name indicates, the pupil makes a choice from the three or more answers which are given. In one form of this test, the word or number which correctly completes a sentence is selected from several words or numbers, usually three, four, or five. In another form, a sentence is completed by choosing one of a number of phrases or clauses which are given. The latter is sometimes called the "best answer" type.

The following are examples of the first form:

1. The number of feet in a mile is 1728 5280 2500 144.
2. $1\frac{3}{4}$ is equal to $\frac{4}{13}$ $1\frac{3}{4}$ $3\frac{1}{4}$ 8.
3. 17% is equal to 17 .17 1700 1.7.
4. The number of square feet in a rectangle 6 feet long and 4 feet wide is 10 20 12 24.

In each of these four items three incorrect responses, as well as the correct response, are suggested. These incorrect responses are not just random numbers but are usually responses which might reasonably be given by one whose information on the subject is vague and indefinite. In the fourth item, for example, the first answer suggested is the sum of the two dimensions, the second is the perimeter, and the third is twice the length. Of course, the correct response may be the first, the second, the third, or the fourth response suggested. Each of these should be the correct response about equally often and, again, there should be no systematic order as to which response is correct.

Examples of the second form are:

1. To find the area of a rectangle, we (a) multiply the length by the width (b) find the distance around it (c) add the two dimensions together.
2. To change common fractions to decimals, we (a) reduce them to lowest terms (b) find the least common denominator (c) divide the numerators by the denominators.
3. The margin on a sale is (a) the selling price plus the cost (b) the selling price minus the cost (c) the selling price times the cost.

Neither form of the multiple response test is as well adapted to arithmetic as to other elementary school sub-

jects. However, its use as a variety form adds interest to the examination and gives the pupil valuable experience in rejecting incorrect statements and selecting those which are correct. When it is used, duplicated copies should be supplied to the pupils. When the test is scored, the pupil usually is given credit for the number of items which he has correct. In the first form, the pupil draws a line under the response which he selects; in the second, he writes in front of the test item the letter indicating his selection.

Matching tests. In a matching test, two series of items are given and the pupil is to decide which item of the second series is best associated with each item of the first series. It is convenient to number the items of the first series and to letter the items of the second series. All that the pupil has to do, then, is to write before or after each item of the first series the letter of the corresponding item of the second series. The pupil's score is the number of correct choices. It is well to have more items in the second series than in the first so that the pupil who knows nearly all of the items will be less likely to guess correctly the few items which he does not know. The following test in denominate numbers will illustrate this type of examination.

Care must be taken in constructing a matching test lest an item in the second list match equally well two items in the first list. For example, since we have "Two pecks" in the first list, we can not have "A half-bushel" for any item which matches one of these will also match the other.

The matching test can also be used for (a) plane

FIRST LIST	ANSWERS	SECOND LIST
1. One bushel	______	(a) Three quarts
2. Two days	______	(b) 1,600 square rods
3. A half-gallon	______	(c) 24
4. Three weeks	______	(d) Ten cents
5. A quart	______	(e) One week
6. One yard	______	(f) Two bushels
7. A mile	______	(g) 18 inches
8. Ten acres	______	(h) 30 days
9. Two pecks	______	(i) 365 days
10. Sixty seconds	______	(j) $16\frac{1}{2}$ feet
11. A half-yard	______	(k) Nine square feet
12. Two dozen	______	(l) 48 hours
13. Thirty minutes	______	(m) Two pints
14. One year	______	(n) 31 days
15. Four feet	______	(o) 320 rods
16. One square yard	______	(p) Four pints
17. One dime	______	(q) One minute
18. Twelve months	______	(r) Four pecks
19. November	______	(s) One year
20. December	______	(t) Three feet
		(u) 144 square inches
		(v) 48 inches
		(w) A half-hour
		(x) 21 days
		(y) 16 quarts
		(z) 366 days

figures (drawings with their dimensions given) and their areas; (b) types of plane figures and formulas for finding their areas; (c) definitions and illustrative cases; (d) common fractions and their decimal equivalents; (e) per cents and their common fraction equivalents (particularly aliquot parts); etc. However, there

are many phases of arithmetic to which the matching test is not well adapted.

Completion exercises. In a completion test exercise, sentences are given with blank spaces in which the pupil is to supply missing words. Such exercises are not easy to prepare for deciding which words should be omitted requires good judgment. The best plan seems to be to prepare a number of statements covering the important items or points of the subject and then to erase certain significant words. The words to be supplied must be significant words and the sentences should not be written so as to suggest what the correct words are. The omitted words should be such that only one answer can be counted correct in each case. The sentence, "There are ____ ______ in a bushel," would not do at all for the blanks are correctly filled by "four pecks," or "32 quarts," or "64 pints."

The following completion test exercise will illustrate the use of this type of examination as it may be applied to certain phases of work with fractions.

1. We reduce a fraction to lower terms by .... both the numerator and the denominator by the same number.
2. When the divisor is .... than one, the quotient is larger than the ....
3. To change an .... fraction to a mixed number, ... the numerator by the ....
4. When the divisor is more than one, the .... is smaller than the ....
5. In a proper fraction, the .... is smaller than the ....

Scores on completion tests correlate highly with scores on intelligence tests. A pupil's success on such a test, then, may be due as much to his general native ability as to his knowledge of the subject matter which the test is designed to cover. Sometimes, the items of a completion test are so difficult that the pupil is unable to see what words belong in the blank spaces even though he may know the facts which the statements demand. Completion tests must be prepared with great care and must be used discreetly.

The use of new-type tests in arithmetic. The samples of tests which have been given indicate the kind of efforts which are being made to remove the defects of old-style examinations. The new-type examinations are far more objective than the examinations of the older kind, they usually have greater validity and reliability, and they are easier to give and to score. However, new-type examinations have their faults. They possess to a lesser degree many of the same defects which have been noted in examinations of the older kind. Furthermore, there is a danger that they may be constructed so as to emphasize specific facts which may have been memorized rather than principles which the pupil should understand. Teachers should work hard on the preparation of new-type tests and should use them for the distinct advantages which they possess.

However, it is not to be expected that the old-style examination question will be entirely displaced. It will be retained but it will be improved so as to eliminate some of its defects and reduce others. There is a value in the experience which children get from *discussing, describing,* and *stating.* The old-style examination prob-

ably is more valuable for the training it gives children in written expression than as a measuring device. In arithmetic, however, there is little to be discussed, described, or stated. The new-type examinations can be used to find out how well the children know the significant facts and, to a degree, to find out how well they understand the fundamental principles and how extensively they have mastered the techniques. But there is no complete substitute in an arithmetic examination for the actual solving of problems and examples.

In choosing and arranging examples and problems for examination purposes, care should be taken to represent the various specific skills which it has been the object of the course to develop. In a fourth grade test in the addition of integers, for instance, each of the various types of addition skills which were discussed in Chapter 2 should be included. Such a test not only yields a fairly accurate general measure of the pupil's ability in addition but becomes a valuable diagnostic device as well, showing, for each pupil, the specific elements which are in need of further practice. Detailed suggestions as to the composition of the topics in the arithmetic of the intermediate grades have been given in the preceding chapters.

Standardized tests. A standardized test is one for which standard scores for the various grades or ages are available. Standard scores are obtained by giving a test to many pupils representing a wide variety of educational and other conditions—urban and rural, East and West, North and South, native-born and foreign born, etc. The standard for a given grade or age is usually the average of the scores of all the children of

that grade or age who have taken the test. A standardized test may be made up of a number of items of equal difficulty—as examples in addition—in which case the standard is the *number* of such examples solved by the average child of a given age or grade in a designated period of time. Or, the test may be composed of items of gradually increasing difficulty, in which case the standard may be the difficulty level which the average pupil is able to reach. Standardized tests are prepared with greater care than are most tests and are frequently criticized by a number of competent judges before they are given to pupils. If the test is made up of items of gradually increasing difficulty, the difficulty of these items is first determined by giving them to pupils and finding the per cent of pupils able to do each item correctly. The items finally selected for the test are then arranged in order of increasing difficulty. Such a test is frequently called a *scale*.

Other standardized tests in arithmetic are mixtures of the two types described in the preceding paragraph. Some tests include several sections representing various phases of arithmetic. The items of each section may be arranged in order of increasing difficulty.

Standardized tests are sometimes classified as general survey tests, diagnostic tests, and standardized practice exercises. General survey tests, as the name indicates, are tests which give a general appraisal of what the pupils have learned. Diagnostic tests indicate more specifically the weaknesses of the individual pupils. They point the way to a program of remedial instruction. Standardized practice exercises are carefully constructed, standardized drill materials.

Advantages of standardized tests. Of course, a well-prepared test is better than a poorly prepared test. Since standardized tests are usually prepared by experts and are prepared with greater care than are most home-made tests, they are better measuring instruments. Ordinarily, they are objective, are easily given and scored, are fairly reliable, and, if they accord with the program of instruction in the schools in which they are used, are probably valid.

A standardized test enables a teacher to tell where her pupils stand with reference to pupils in the nation who belong to the same grade or age group. It is distinctly worth while to know whether a class can add fractions as well as or better than the average class of the same grade. In one small city where the author directed a testing program, it was found that the average attainment of the eighth-grade pupils in addition, subtraction, multiplication, and division of integers was just equal to the fifth-grade standard. This information came as a distinct shock to the three eighth-grade teachers concerned. They could hardly believe that their pupils were relatively so inefficient in these processes. It was too late to do much for the eighth-grade pupils but a program of remedial teaching was at once undertaken in the grades below the eighth, each of which was also much below standard.

Not only does a standardized test tell the teacher where her pupils stand but it also enables each pupil to see where he stands. He is shown his position in the group to which he belongs and also his position with reference to the standard scores. The author has in mind an over-age fourth grade boy who was so far below the

grade norms in all of the processes of arithmetic that his teacher believed that he would be completely discouraged if he were shown his standing. The effect, however, was just the opposite. For the first time in his school life he began to take his practice work in arithmetic seriously and to show an interest in more efficient methods. He learned to plot a graph showing his progress from week to week and by the end of the year had almost reached the standard for his grade. He made more progress than was made by any other pupil in the school.

Standardized tests thus furnish a goal for both the teacher and the pupils. Setting a goal for the month, for the semester, or for the year supplies a motive for work and it is well known that an appealing motive is a powerful stimulus for effective work. It is much more interesting to work toward some definite goal than just to work. There are other ways of supplying motives, to be sure, but there are few which act more readily and more surely than setting up specific goals to be reached.

Standardized tests also make possible certain interesting and important comparisons. We can compare the pupils in the class, discover the great variability which usually exists, and plan instruction which shall be differentiated for the various ability levels. We can compare the class with other classes of the same grade in the same school system. Interesting group rivalry is sometimes fostered thereby. We can compare the classes of one grade with the classes of the grades which precede and follow and ascertain the average amount of improvement from grade to grade. And, as already indi-

cated, we can compare attainments with average attainments for the nation.

When used in conjunction with intelligence tests, standardized tests make it possible to evaluate a pupil's performance in terms of his ability. If a ten-year-old boy has a mental age of ten years (I. Q., 100) he is of average ability and can be expected to do average work; all that he attains beyond what average ten-year-olds attain is just that much extra to his credit. But if he has a mental age of eight years (I. Q., 80) we cannot expect him to do so well as other ten-year-olds. If, on the other hand, his mental age is twelve years (I. Q., 120) a ten-year standard is below his ability level and we are justified in expecting him to make greater progress. Evaluating the performances of pupils in terms of their native ability throws new light on problems of instruction, problems of classification, problems of promotion, and, sometimes, problems of discipline. It is usually found that the dullest children are the most accelerated, when progress is measured in terms of mental age.

Disadvantages of standardized tests. There has been a tendency for some teachers to look upon standardized tests as educational panaceas which not only would eliminate the faults that other tests were known to possess but also would be free of faults themselves. However, students of tests and testing have witnessed many educational crimes which were committed through the misuse of tests and test results. In other words, standardized tests may have their disadvantages as well as their advantages.

In the first place, the tests themselves may be faulty. They are usually objective but they may be unreliable

and, hence, invalid. Before purchasing standardized tests, teachers should insist upon seeing the statistical evidence which has been collected as to the validity and the reliability of the tests.

Secondly, the use of standardized tests sometimes has too great an influence upon schoolroom practices. If teachers know that their pupils are to be tested and if they have some information as to the nature of the tests which are to be used, they may teach so as merely to prepare their pupils for the tests. If the tests measured all of the desirable educational outcomes, getting ready for tests would not be a bad method of teaching; but it is readily admitted that some desirable educational outcomes are not measured by the standardized tests which are used. Furthermore, the use of standardized tests sometimes has the effect of perpetuating the *status quo*. It is useless to undertake to rid the course of unreal problems, for instance, if the standardized tests which are used contain such problems.

In Chapter One, it was suggested that the use of standardized tests sometimes had the effect of encouraging the use of methods which are based upon the drill theory of instruction. It is easily possible for a testing program, standardized or otherwise, to cause teachers to think too much of those outcomes of arithmetic teaching which are easily measured. Good tests are a most valuable aid in teaching arithmetic; but poor tests, or good tests poorly used, may easily do more harm than good by focusing attention on routine skills to the exclusion of more worthwhile objectives. In testing, as elsewhere, the meaning theory should predominate. Tests should measure the pupil's understanding of basic ideas and his grasp of

fundamental principles as well as his skill in the processes. There is an increasing tendency to do this in the newer and better standardized tests.

Again, teachers are sometimes wrongly judged by the performances of their pupils on tests. There are many factors which contribute to the success of teachers and not nearly all of these are measured by standardized tests. A teacher may very properly use standardized tests as one means of evaluating the results of her efforts but the sometimes hasty judgments which supervisors base upon test results may easily be wrong. The practice of basing promotions and salary increases of teachers upon the scores which pupils make on standardized tests may drive teachers to the use of methods which are founded on the drill theory and to a very narrow interpretation of the aims of education.

Finally, tests often are misused in that the results are not carefully analyzed and made the basis for improvement of instruction. Many dollars and many hours of time have been expended on tests which were merely filed away in an office somewhere and which failed almost entirely to affect teaching practices.

An intelligent and fairminded use of tests will avoid most of the disadvantages which have been enumerated. Hence, standardized tests should be used for the benefits which have been described. It is a good plan to use them at least three times in the school year: first, soon after the beginning of the year; again, near the middle of the year; and finally, near the end of the year.

Standardized tests in arithmetic. Standardized tests in arithmetic and other elementary school subjects began to appear in the years, 1909 to 1915. Since the latter

date, a great many arithmetic tests have been published. Generally speaking, the more recently published tests are better than the older tests, although there are exceptions.

Before purchasing tests, teachers should examine the statistics as to validity and reliability, as has already been suggested, and also should find out the grades for which the tests are designed, whether they are general survey or diagnostic, whether they have been standardized, and the prices.

More arithmetic tests are published by the Public School Publishing Company at Bloomington, Illinois, than by any other concern. Tests are published also by The World Book Company, Yonkers-on-Hudson, New York; by The Educational Test Bureau, Minneapolis, Minnesota; by the Bureau of Publications, Teachers College, Columbia University, New York City; by the Southern California School Book Depository, Ltd., Hollywood, California; by Houghton Mifflin Company, Boston, Massachusetts; by Scott, Foresman and Company, Chicago, Illinois; by the Bureau of Educational Measurements and Standards, Kansas State Teachers College, Emporia, Kansas; and by others.

Locally constructed tests are used for state-wide testing programs in several states, such as Iowa, Kansas, and Ohio. These tests are usually prepared by well qualified persons and, after use, are standardized for the states in which they are used. Usually, such state tests may be obtained by interested persons and at a very reasonable price.

Testing is an important phase of teaching. Undoubtedly, examinations have been poorly constructed and

administered and the results have been interpreted unwisely. However, it has been shown that the defects of examinations can be eliminated or considerably reduced. Taking an examination should be a beneficial experience to a pupil. It should be a valuable review, helping him to organize the facts which he has learned and to retain them. Rather than abolish the examination, then, we should keep it and improve it. Its judicious use should mean better teaching and better learning.

QUESTIONS AND REVIEW EXERCISES

1. What are the more common defects in old-style examinations? Can these defects be eliminated or materially reduced?

2. Should an examination on a unit of work cover all of the important points in the unit? If the examination covers only one-fourth of the important points in the unit, is it possible that a pupil who knows three-fourths of the points in the unit may score zero on the test? Is it possible that a pupil who knows only one-fourth of the points may make a perfect score on the test? Is either of these probable?

3. Should final examinations be prepared by the teachers or by supervisors, principals, or others? Summarize the advantages and the disadvantages of each plan.

4. Under what circumstances is rate of work an important dimension of a pupil's ability?

5. Should those who stand high in a subject be excused from final examinations? What are the advantages and the disadvantages of such a plan?

6. What benefits is it reasonable to expect from the use of well prepared and properly administered examinations?

7. What is the difference between relative ability and absolute ability?

8. Which do you think should require the more time and energy from the teacher, the preparation of an examination or the preparation for an ordinary class period?

9. The examination sometimes is referred to as the examination lesson. May an examination be a lesson?

10. Give suggestions for preventing cheating on examinations. What would you do about cheating if you detected it?

11. What is meant by objectivity in examinations? How may an examination be made objective?

12. Does it pay to bluff on an old-style examination? On a new-style examination?

13. How would you reduce the subjectivity of an essay examination?

14. When is an examination valid? Is it correct to say that the value of an examination varies directly with its validity?

15. How can a teacher increase the validity of an examination? How is the validity of an examination measured?

16. When is an examination reliable? Can an examination be reliable but not valid? Can it be valid but not reliable?

17. How is the reliability of an examination determined? How can reliability be increased?

18. Which is easier to prepare, an old-type examination or a new-type examination? Which is easier to administer? Which is easier to score? Which is easier for pupils to take?

19. Which would you prefer, to spend one hour preparing an examination and three hours scoring the papers or to spend three hours preparing an examination and one hour scoring the papers?

20. What are the chief kinds of new-type examinations?

21. Summarize the directions for preparing alternate response tests. How should they be scored?

22. What are the more common faults of true-false tests? How can these faults be eliminated?

23. What is a multiple response test? How many answers should be supplied in such a test? How should these answers be chosen?

24. Why is it customary to use the right-minus-wrong method in scoring true-false tests but to give credit for the number right in scoring a multiple response test?

25. Give directions for preparing a matching test. Why should the number of items in the two columns not be equal?

26. Give directions for preparing completion tests. Why is it harder to prepare completion tests than other new-type tests? Do pupils find completion tests harder than other new-type tests?

27. Is it reasonable to expect all arithmetic tests to be new-type tests? Why?

28. What is a standardized test? Why do you think arithmetic was one of the first school subjects for which standardized tests were developed?

29. How is the standard on a standardized test determined? What is a *norm?*

30. What can you say as to the progress of a class if one-half of the pupils are above the standard and one-half are below it?

31. What is the difference between a test and a scale?

32. Are standardized tests usually better than home made tests? Why?

33. What is the difference between a general survey test and a diagnostic test? Under what conditions should each be used?

34. What advantages may the teacher expect to gain from the use of standardized tests?

35. What is meant by mental age? Chronological age? I. Q.? Of what use are intelligence tests in evaluating the performances of pupils on achievement tests?

36. Should the work of a teacher be judged entirely in terms of the performance of her pupils on standardized tests? Why?

37. Summarize the disadvantages which may arise from the use of standardized tests in arithmetic.

38. Can a test be reliable without being objective?

CHAPTER TEST

Determine whether each statement is true or false. A scoring key will be found on page 534.

1. By absolute ability is meant the amount of ability of an individual above zero ability.

2. By relative ability is meant the ability of a pupil with reference to other members of the class.

3. Examinations should supply a motive for better work.

4. Preparation for an examination should require less time of the teacher than preparation for the average of other lessons.

5. If two persons grade a perfectly objective examination paper, they will obtain the same score.

6. The scoring of objective examinations requires good judgment on the part of the scorer.

7. Objective examination questions sometimes begin with the word "Describe."

8. Bluffing pays on a subjective examination more than on an objective examination.

9. In the long run, it makes no difference whether a pupil guesses on items he does not know on a true-false test or omits them.

10. The essay type of examination is usually objective.

11. Variability in scoring subjective examinations can be considerably reduced by the use of a set of scoring rules.

12. Validity and reliability are synonymous terms.

13. Validity in an examination is more important than objectivity.

14. If an examination is valid, it must also be reliable.

15. If an examination is reliable, it must also be valid.

16. If an examination is reliable, it must measure what it is supposed to measure.

17. The reliability of a test is measured by correlating the scores with some acceptable criterion.

18. An objective examination usually is more reliable than is a subjective examination.

19. Other things being equal, the longer a test the greater its reliability.

20. Essay examinations usually require more time for preparation than do new-type examinations.

21. Essay examinations usually require more time for scoring the papers than do new-type examinations.

22. A true-false test is an alternate response test.

23. In a true-false test, the true statements and the false statements should be arranged in a systematic order.

24. Some alternate response test items should be ambiguous.

25. On an alternate response test, the score is usually the number right.

26. On a multiple response test, the score is usually the number right.

27. Multiple response tests are as well adapted to arithmetic as to other elementary school subjects.

28. In a matching test, the number of items in the second series should be equal to the number of items in the first series.

29. Scores on completion tests usually correlate highly with scores on intelligence tests.

30. It is a reasonable expectation that the old-style examination question will be entirely displaced.

31. A pupil who just reaches the standard on a standardized test is better than the average.

32. Standardized tests are prepared with greater care than are most tests.

33. In a scale, the items are all of equal difficulty.

34. A diagnostic test is designed to give a general overview of what has been learned.

35. The I. Q. is the ratio of the mental age to the chronological age.

36. The use of standardized tests sometimes tends to retard educational progress.

37. Standardized tests should be used at least three times in the school year.

38. Standardized tests always should be administered to pupils by supervisors, principals, or superintendents.

39. Those pupils having the highest grades should be excused from final examinations.

40. The judicious use of examinations should mean better teaching and better learning.

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2. Kelly, Frederick James. *Teachers Marks: Their Variability and Standardization.* New York: Teachers College, Columbia University, 1914, 139 pp. This is an early but very able discussion of teachers marks. Many interesting data are presented.

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Chapter XII, pages 140-146. The use of scientifically constructed tests is ably treated in Chapter XIV.

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ANSWERS FOR CHAPTER TESTS

CHAPTER 1. 1. False. 2. True. 3. True. 4. False. 5. True. 6. False. 7. True. 8. True. 9. True. 10. False. 11. True. 12. True. 13. False. 14. False. 15. True. 16. False. 17. False. 18. False. 19. True. 20. False. 21. True. 22. True. 23. False. 24. False. 25. False. 26. False. 27. True. 28. False. 29. True. 30. True.

CHAPTER 2. 1. (1). 2. (2). 3. (2). 4. (1). 5. (3). 6. (1). 7. (3). 8. (3). 9. (3). 10. (1). 11. (3). 12. (2). 13. (2). 14. (3). 15. (1). 16. (3). 17. (2). 18. (1). 19. (3). 20. (2). 21. (3). 22. (1). 23. (1). 24. (3). 25. (3). 26. (1). 27. (1). 28. (2). 29. (2). 30. (1). 31. (3). 32. (2). 33. (2). 34. (1). 35. (1). 36. (3). 37. (2). 38. (1). 39. (1). 40. (3).

CHAPTER 3. 1. True. 2. True. 3. False. 4. False. 5. True. 6. True. 7. True. 8. False. 9. False. 10. False. 11. True. 12. True. 13. False. 14. True. 15. True. 16. True. 17. True. 18. False. 19. False. 20. False. 21. True. 22. False. 23. False. 24. False. 25. True. 26. False. 27. True. 28. False. 29. True. 30. False. 31. False. 32. False. 33. False. 34. True. 35. False. 36. False. 37. False. 38. False. 39. True. 40. True.

CHAPTER 4. 1. (3). 2. (3). 3. (1). 4. (3). 5. (1). 6. (2). 7. (2). 8. (3). 9. (1). 10. (2). 11. (1). 12. (2). 13. (2). 14. (3). 15. (1). 16. (1). 17. (3). 18. (2). 19. (2). 20. (1). 21. (1). 22. (3). 23. (3). 24. (2). 25. (2).

CHAPTER 5. 1. (2). 2. (3). 3. (2). 4. (3). 5. (1). 6. (1). 7. (2). 8. (1). 9. (3). 10. (1). 11. (3). 12. (2). 13. (2). 14. (1). 15. (3). 16. (1). 17. (3). 18. (3). 19. (2). 20. (1). 21. (2).

CHAPTER 6. 1. True. 2. True. 3. False. 4. True. 5. False. 6. False. 7. False. 8. False. 9. False. 10. False. 11. True. 12. True. 13. True. 14. True. 15. True. 16. True. 17. True. 18. False. 19. False. 20. True. 21. False. 22. True. 23. False. 24. False. 25. True. 26. True. 27. False. 28. True. 29. True.

30. False. 31. True. 32. True. 33. False. 34. True. 35. False.

CHAPTER 7. 1. (2). 2. (3). 3. (1). 4. (1). 5. (3). 6. (2). 7. (3). 8. (3). 9. (1). 10. (2). 11. (2). 12. (1). 13. (3). 14. (1). 15. (2). 16. (2). 17. False. 18. True. 19. True. 20. True. 21. True. 22. True. 23. False. 24. False. 25. True. 26. True. 27. False. 28. True. 29. False. 30. False. 31. False. 32. False.

CHAPTER 8. 1. (2). 2. (3). 3. (2). 4. (2). 5. (1). 6. (3). 7. (3). 8. (1). 9. (3). 10. (2). 11. (1). 12. (1). 13. (3). 14. (2). 15. (1). 16. (3). 17. (1). 18. (1). 19. (2). 20. (3). 21. (2). 22. (3). 23. (2). 24. (1).

CHAPTER 9. 1. True. 2. True. 3. False. 4. True. 5. True. 6. False. 7. False. 8. False. 9. True. 10. False. 11. True. 12. False. 13. False. 14. False. 15. True. 16. True. 17. False. 18. (h). 19. (d). 20. (l). 21. (a). 22. (b). 23. (n). 24. (j). 25. (c). 26. (o). 27. (e). 28. (g). 29. (q). 30. (k).

CHAPTER 10. 1. False. 2. False. 3. False. 4. True. 5. True. 6. False. 7. False. 8. True. 9. True. 10. False. 11. False. 12. True. 13. True. 14. True. 15. False. 16. True. 17. (f). 18. (d). 19. (a). 20. (g). 21. (b). 22. (j). 23. (c). 24. (e). 25. (k). 26. (o).

CHAPTER 11. 1. (2). 2. (2). 3. (2). 4. (1). 5. (1). 6. (3). 7. (1). 8. (1). 9. (1). 10. (2). 11. (2). 12. (3). 13. (1). 14. (3). 15. (3). 16. (3). 17. (3). 18. (1). 19. (3). 20. (1).

CHAPTER 12. 1. True. 2. True. 3. True. 4. False. 5. True. 6. False. 7. False. 8. True. 9. True. 10. False. 11. True. 12. False. 13. True. 14. True. 15. False. 16. False. 17. False. 18. True. 19. True. 20. False. 21. True. 22. True. 23. False. 24. False. 25. False. 26. True. 27. False. 28. False. 29. True. 30. False. 31. False. 32. True. 33. False. 34. False. 35. True. 36. True. 37. True. 38. False. 39. False. 40. True.

INDEX